CHAPTER 4

# TRANSLATIONAL DYNAMICS

*In the next few notes we study the laws that tell us how objects move when forces and torques act on them. This note is about forces and Newton's three laws. It also covers momentum, the center of mass, the center-of-mass frame and systems whose moving mass changes, because every later note on dynamics uses them.* [CP1 Ch 4 intro]

## 4.1 LINEAR MOMENTUM

[CP1 4.1]

We should not talk about force on its own. We first need the idea of *linear momentum*. The linear momentum $\vec{p}$ of a particle is

$$\vec{p} = m\vec{v},$$

where $m$ is the particle's mass and $\vec{v}$ is its velocity. The total momentum of a system of particles is just the sum of the momenta of all its particles.

This definition comes from an interesting fact found by experiment: under certain conditions, the total momentum of a system is conserved (it does not change). We come back to this in Section 4.4. For now, just remember that linear momentum is a useful quantity with some special properties.

## 4.2 NEWTON'S THREE LAWS

[CP1 4.2]

### 4.2.1 The First Law and Inertial Frames

[CP1 4.2.1]

> **Newton's first law.** *An object keeps moving at constant velocity unless a net external force acts on it.*

The first law actually *defines* an inertial frame of reference: an *inertial frame* is a frame in which the first law is true. An inertial frame has no absolute acceleration. That is, an accelerometer placed in that frame measures zero acceleration.

In Newtonian physics, acceleration is absolute, while velocity is relative. Here is a set-up similar to Newton's famous thought experiment with a rotating bucket of water. You hold a glass of water on a train. When the train moves at constant velocity, the water surface stays flat. When the train accelerates, the surface tilts, and the tilt shows how large the acceleration is. Every outside observer, whether accelerating or moving at constant velocity, sees the same tilt. So all of them agree that you have an acceleration of one definite size. In this sense, Newtonian physics has an absolute acceleration.

We can add to the definition above. Inertial frames form a set of frames that move at constant velocity relative to each other. That is, a frame that moves at constant velocity relative to an inertial frame is also an inertial frame. Newton assumed that there is an *absolute space* that is truly at rest. He used the frame of the distant stars as the reference from which all other inertial frames can be found.

In the end, the first law sets the context in which Newton's laws are valid. It is *not* a special (limiting) case of the second law; this is a common misconception. To see why, note that there are frames in which free particles (particles with no force on them) accelerate. For example, passengers in an accelerating train see the world outside accelerate. This seems to break Newton's second law, but there is no problem: Newton's laws cannot be used in this frame in the first place, because it is not an inertial frame.

### 4.2.2 The Second Law

[CP1 4.2.2]

> **Newton's second law.** *The net external force* $\sum \vec{F}$ *on a particle equals the rate of change of its momentum* $\vec{p}$*:*

$$\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}. \tag{4.1}$$

Substituting $\vec{p} = m\vec{v}$ for a particle (its mass $m$ is constant),

$$\sum \vec{F} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = m\vec{a}, \tag{4.2}$$

where $\vec{a}$ is the acceleration of the particle.

A similar $\sum \vec{F} = m\vec{a}$ equation holds for a *system* of particles, but there $\vec{a}$ means a different physical quantity. This is to be expected: we cannot just put the acceleration of one chosen particle into the equation. We derive this form in Section 4.3, after the third law.

### 4.2.3 The Third Law

[CP1 4.2.3]

When you punch a wall, the wall "hits" you back: you feel the impact on your knuckles. The third law states this exactly:

$$\vec{F}_{AB} = -\vec{F}_{BA}, \tag{4.3}$$

where $\vec{F}_{AB}$ is the force on object $A$ by object $B$. So when $A$ pushes or pulls on $B$, $B$ pushes or pulls on $A$ with a force of equal size in the opposite direction. The first two laws say nothing about how the *source* of a force moves; the third law brings the source into the picture.

Keep in mind that the third law has exceptions, such as the magnetic force. For example, take two positive charges moving along the $x$- and $y$-axes. The magnetic force on each charge due to the other points in a different direction (along $y$ for one and along $x$ for the other), so the two forces are not equal and opposite. Linear momentum is still conserved, however, if we give the electromagnetic field a momentum of its own.

The third law as written above is called the *weak law of action and reaction*. The *strong law* also requires the two equal and opposite forces to act along the same line. Most forces, such as gravity and the normal force, obey the strong law.

## 4.3 NET EXTERNAL FORCE ON A SYSTEM OF PARTICLES

[CP1 4.3]

Newton's second and third laws together give a neat form of the second law for a system of particles. Take a system of $N$ particles. Let $\left(\sum \vec{F}\right)_i$ be the net *external* force on the $i$th particle, and let $\vec{f}_{ij}$ be the *internal* force on the $i$th particle due to the $j$th particle. The second law for the $i$th particle is

$$\left(\sum \vec{F}\right)_i + \sum_{j,\, j\neq i} \vec{f}_{ij} = \frac{\mathrm{d}\vec{p}_i}{\mathrm{d}t}.$$

Adding these equations for all particles,

$$\sum_{i=1}^{N}\left(\sum \vec{F}\right)_i + \sum_{i,j,\, i\neq j} \vec{f}_{ij} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} \vec{p}_i\right). \tag{4.4}$$

The first term is the total net external force on the system, $\sum \vec{F}_{\text{ext}}$. The second term is zero, because by the third law

$$\vec{f}_{ij} = -\vec{f}_{ji},$$

so the internal forces cancel in pairs. Equation (4.4) becomes

$$\sum \vec{F}_{\text{ext}} = \frac{\mathrm{d}\vec{P}}{\mathrm{d}t} = \sum_{i=1}^{N} m_i \vec{a}_i, \tag{4.5}$$

where $\vec{P} = \sum_i \vec{p}_i$ is the total momentum. In words: the rate of change of the total momentum of a system, $\mathrm{d}\vec{P}\big/\mathrm{d}t$, equals the net external force on the system.

Now define a point that describes where the system is as a whole. It is called the *center of mass*, and its position vector is

$$\vec{R} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}, \tag{4.6}$$

where $\vec{r}_i$ is the position vector of the $i$th particle. The total mass of the system is

$$M = \sum_{i=1}^{N} m_i.$$

Multiplying Eq. (4.6) by $M$ and differentiating with respect to time gives

$$\vec{P} = \sum_{i=1}^{N} m_i \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} = M\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = M\vec{v}_{\text{CM}}, \tag{4.7}$$

where $\vec{v}_{\text{CM}} = \mathrm{d}\vec{R}\big/\mathrm{d}t$ is the velocity of the center of mass. Putting this into Eq. (4.5),

$$\sum \vec{F}_{\text{ext}} = M\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} = M\vec{a}_{\text{CM}}, \tag{4.8}$$

where $\vec{a}_{\text{CM}} = \mathrm{d}^2\vec{R}\big/\mathrm{d}t^2$ is the acceleration of the center of mass.

So, in its translational motion, a system of particles responds to a net external force as if it were a single imaginary mass $M$ placed at the center of mass $\vec{R}$. Equation (4.8) is very important for a rigid body. We no longer need to write $\vec{F} = m\vec{a}$ for every particle. We only need the translational motion of the center of mass, and this gives the translational motion of the whole body, because the distances between its particles do not change. (We still need to study the rotation about the center of mass, by other methods, to find how the body is oriented; see Note 5.)

For a continuous mass distribution, the same steps lead to

$$\vec{\boldsymbol{R}} = \frac{\int \vec{\boldsymbol{r}}\,\mathrm{d}m}{\int \mathrm{d}m}, \tag{4.9}$$

where $\vec{\boldsymbol{r}}$ is the position vector of the infinitesimal mass element $\mathrm{d}m$. That is, $\int \vec{\boldsymbol{r}}\,\mathrm{d}m$ means: take the position vector $\vec{\boldsymbol{r}}$ of each tiny mass element, weight it by that element's mass $\mathrm{d}m$, and add over the whole distribution. This can be a line, surface or volume integral (Note 1 shows how to set up such elements).

## 4.4 CONSERVATION OF LINEAR MOMENTUM AND IMPULSE

[CP1 6.1]

***Conservation of linear momentum.*** Recall from Eq. (4.5) that the net external force on a system equals the rate of change of its total momentum,

$$\sum \vec{\boldsymbol{F}} = \frac{\mathrm{d}\vec{\boldsymbol{p}}}{\mathrm{d}t}.$$

If the net external force on the system is zero,

$$\frac{\mathrm{d}\vec{\boldsymbol{p}}}{\mathrm{d}t} = 0 \quad \Longrightarrow \quad \vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_0$$

for some constant momentum $\vec{\boldsymbol{p}}_0$.

> **Conservation of linear momentum.** *The total linear momentum of a system is conserved if no net external force acts on it.*

***Impulse-momentum theorem.*** We get another view of the second law by separating variables and integrating from time $t_0$ (momentum $\vec{\boldsymbol{p}}_0$) to time $t_1$ (momentum $\vec{\boldsymbol{p}}_1$):

$$\int_{t_0}^{t_1} \left(\sum \vec{\boldsymbol{F}}\right) \mathrm{d}t = \int_{\vec{\boldsymbol{p}}_0}^{\vec{\boldsymbol{p}}_1} \mathrm{d}\vec{\boldsymbol{p}}\,.$$

Define the *impulse* delivered by an external force $\vec{\boldsymbol{F}}_i$ between times $t_0$ and $t_1$ as

$$\vec{\boldsymbol{J}} = \int_{t_0}^{t_1} \vec{\boldsymbol{F}}_i\,\mathrm{d}t\,. \tag{4.10}$$

Then the equation above becomes

$$\sum \vec{\boldsymbol{J}} = \Delta\vec{\boldsymbol{p}}. \tag{4.11}$$

> **Impulse-momentum theorem.** *The change in the linear momentum of a system during a time interval equals the total impulse delivered to the system during that interval.*

This form is very useful for *impulsive forces*, such as the normal forces between colliding particles. They are huge but act only for a very short time. Their "job" is to give a system momentum over a short time interval.

## 4.5 CENTER OF MASS

[CP1 4.3.1, 6.5, 6.5.2]

We now show how to find the center of mass of a system, with a few examples.

### 4.5.1 Discrete Particles and the Grouping Trick

[CP1 4.3.1]

Start with a simple system of two particles, $m$ and $M$, at coordinates $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$. From Eq. (4.6),

$$\vec{\boldsymbol{R}} = \frac{1}{m+M}\begin{pmatrix} mx_1 \\ my_1 \\ mz_1 \end{pmatrix} + \frac{1}{m+M}\begin{pmatrix} Mx_2 \\ My_2 \\ Mz_2 \end{pmatrix} = \frac{1}{m+M}\begin{pmatrix} mx_1 + Mx_2 \\ my_1 + My_2 \\ mz_1 + Mz_2 \end{pmatrix}.$$

***The grouping trick.*** The formula for $\vec{\boldsymbol{R}}$ is linear (it is a weighted sum). This gives a useful trick for a system of discrete particles, or for a system made of several continuous mass distributions. Pick a few particles or distributions. Find their center of mass, and replace them by one point mass, equal to their total mass, placed exactly there. Then find the center of mass of the whole system, with this point mass in place of the particles it replaced. (The proof is left as an exercise.)

***Symmetry.*** The center of mass of a uniform mass distribution must lie on every line or plane of symmetry of the distribution. This follows directly from the definition. For example, in two dimensions, put the origin on a line of symmetry and let the $y$-axis lie along this line. Now look at the direction perpendicular to the line (the $x$-direction). By symmetry, for every particle at $(x, y)$ there is an equal particle at $(-x, y)$. So the weighted sum of the $x$-coordinates is zero, and the center of mass lies on the line of symmetry. As this shows, the point where lines or planes of symmetry meet gives the center of mass.

Let us use the grouping trick on a simple case: the center of mass $\vec{\boldsymbol{R}}$ of the two uniform spheres in Fig. 4.1. No clever choice of coordinates turns this into a simple integral with fixed limits. But by symmetry, the center of mass of each sphere is at its center. So we replace the spheres by two point masses, $m$ and $M$, at their centers, and then use the two-particle result above.

**Figure 4.1**

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**Figure 4.1.** Center of mass of two spheres.

***Missing masses.*** The same method works for objects with "missing" mass (holes). First, add suitable imaginary masses to fill the holes. Then replace

- the combined system (the real object plus the imaginary masses) by its total mass placed at its center of mass, and
- the imaginary masses by a *negative* mass of the same size, placed at the center of mass of the imaginary masses.

The center of mass of this new set-up is the same as that of the real object.

**EXAMPLE.** [CP1 4.3.1, p.142] A uniform circle of radius $R$ has a circular hole of radius $\frac{R}{2}$ (Fig. 4.2). The surface mass density (mass per unit area) is $\sigma$. Find the center of mass of the object.

**Figure 4.2**

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**FIGURE 4.2.** Circle with a hole.

***Solution*** First "fill up" the hole. This gives a complete circle, which we replace by a mass $M = \sigma\pi R^2$ at its center. To cancel the mass we added, we must take away the mass of the hole. This is the same as adding a "negative mass" in the hole, that is, a mass

$$-m = -\tfrac{1}{4}\sigma\pi R^2$$

at the center of the hole. (The hole has radius $R/2$, so its area is $\pi R^2/4$.) Put the origin at the center of the complete circle. The center of the hole is a distance $R/2$ from it. So the object with the hole has

$$x_{\text{CM}} = \frac{-m \cdot \frac{R}{2}}{M - m} = -\frac{R}{6}.$$

(Check: $-\frac{1}{4} \cdot \frac{1}{2} \big/ \left(1 - \frac{1}{4}\right) = -\frac{1}{8} \cdot \frac{4}{3} = -\frac{1}{6}$.)

## 4.5.2 Center of Mass of a Continuous Distribution

[CP1 4.3.1]

For a continuous mass distribution we usually need to integrate. The main difficulty is choosing a convenient coordinate system and the correct limits of integration, which must describe the actual shape of the distribution. Note 1 treats these points in detail. The next two examples should give some feel for them.

**EXAMPLE.** [CP1 4.3.1, p.143] Find the center of mass of a uniform right-angled triangle with surface mass density $\sigma$, lying in the $xy$-plane. Its angle of inclination is $\theta$ and its base has length $l$.

**Figure 4.3**

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**FIGURE 4.3.** Center of mass of a triangle.

***Solution*** Use tiny rectangular elements in Cartesian coordinates, with sides d$x$ and d$y$. Then $\mathrm{d}m = \sigma\,\mathrm{d}x\,\mathrm{d}y$. By Eq. (4.9) we need the double integral

$$\vec{\boldsymbol{R}} = \frac{1}{M}\iint_S \begin{pmatrix} x \\ y \end{pmatrix} \sigma\,\mathrm{d}x\,\mathrm{d}y$$

over the surface $S$, which here is the triangle. As always, $M$ is the total mass of the triangle.

Here the limits of integration depend on the order of integration. We choose to integrate over $y$ first and then over $x$. In the diagram, this means we first add up a thin vertical strip of mass at position $x$ (right of Fig. 4.3), and then add up all the strips. For a vertical strip at $x$, $y$ runs from 0 to $x\tan\theta$ (the height of the triangle there). Then $x$ runs from 0 to $l$:

$$\vec{\boldsymbol{R}} = \frac{1}{M}\int_0^l \int_0^{x\tan\theta} \begin{pmatrix} x \\ y \end{pmatrix} \sigma\,\mathrm{d}y\,\mathrm{d}x\,.$$

We do the two coordinates separately. The inner integral over $y$ is easy: for $x_{\text{CM}}$ it gives $x \cdot x\tan\theta$, and for $y_{\text{CM}}$ it gives $\frac{1}{2}(x\tan\theta)^2$. So

$$\begin{aligned} x_{\text{CM}} &= \frac{\int_0^l \int_0^{x\tan\theta} x\sigma\,\mathrm{d}y\,\mathrm{d}x}{M} = \frac{\int_0^l x^2\tan\theta\,\sigma\,\mathrm{d}x}{M} \\ &= \frac{l^3\tan\theta\,\sigma}{3M} = \frac{2l}{3}, \\ y_{\text{CM}} &= \frac{\int_0^l \int_0^{x\tan\theta} y\sigma\,\mathrm{d}y\,\mathrm{d}x}{M} = \frac{\int_0^l x^2\tan^2\theta\,\sigma\,\mathrm{d}x}{2M} \\ &= \frac{l^3\tan^2\theta\,\sigma}{6M} = \frac{l\tan\theta}{3}. \end{aligned}$$

The last step in each line uses the mass of the triangle, $M = \frac{1}{2}l \cdot l\tan\theta \cdot \sigma = \frac{l^2\tan\theta\,\sigma}{2}$.

We could have stopped after finding $x_{\text{CM}}$. The result $x_{\text{CM}} = \frac{2l}{3}$ does not depend on $\theta$. By symmetry (turn the triangle so that its vertical side becomes the base), $y_{\text{CM}}$ must then be one-third of the height of the right-hand tip.

Notice that the center of mass here is where the three *medians* of the triangle meet (a median joins a vertex to the midpoint of the opposite side). This point divides each median into two parts in the ratio 2 : 1, with the shorter part nearer the base. This is true for every triangle and is a well-known result of geometry.

**EXAMPLE.** [CP1 4.3.1, p.144] Find the center of mass of a uniform semi-circle with surface mass density $\sigma$ and radius $R$.

**Figure 4.4**

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**Figure 4.4.** Semi-circle.

***Solution*** Polar coordinates are convenient here. The tiny mass element at polar coordinates $(r, \theta)$ is a small rectangle with sides $\mathrm{d}r$ and $r\,\mathrm{d}\theta$, so $\mathrm{d}m = \sigma r\,\mathrm{d}\theta\,\mathrm{d}r$. Its position vector in Cartesian coordinates is $(r\cos\theta,\ r\sin\theta)$.

Writing the position in *Cartesian* components is the key step. The polar unit vectors change direction from point to point, and integrating them would be very awkward. The Cartesian unit vectors are fixed, so they come out of the integral. So

$$\vec{\boldsymbol{R}} = \frac{1}{M}\int_0^R \int_0^\pi \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \sigma r\,\mathrm{d}\theta\,\mathrm{d}r\,.$$

By symmetry $x_{\mathrm{CM}} = 0$, so we only compute $y_{\mathrm{CM}}$. Using $\int_0^\pi \sin\theta\,\mathrm{d}\theta = 2$,

$$\begin{aligned} y_{\mathrm{CM}} &= \frac{1}{M}\int_0^R \int_0^\pi r^2 \sin\theta\,\sigma\,\mathrm{d}\theta\,\mathrm{d}r = \frac{1}{M}\int_0^R 2r^2\sigma\,\mathrm{d}r \\ &= \frac{2\sigma R^3}{3M} = \frac{4R}{3\pi}, \end{aligned}$$

where the last step uses $M = \frac{1}{2}\pi R^2\sigma$.

## 4.5.3 Rotations

[CP1 4.3.1]

For objects with rotational symmetry, like the semi-circle above, there is another way. Imagine rotating the object by a small angle $\varphi$, so that a little mass moves from one end to the other, and work out the effect. The clearest way to keep track of this is through the change in gravitational potential energy in a uniform gravitational field (Note 6). This is only a bookkeeping device: it comes purely from the change in the coordinates of the masses and has nothing to do with the dynamics of the system.

**Figure 4.5**

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**Figure 4.5.** Rotated sector.

Let us find $y_{\mathrm{CM}}$ of a uniform sector (a “pizza slice”) of angle $2\theta$ and radius $R$. It is placed symmetrically about the $y$-axis, with its tip at the origin. Rotate the sector clockwise by a small angle $\varphi$. The result is the same as moving a thin isosceles triangle (two equal sides $R$, apex angle $\varphi$) from the left edge to the right edge (Fig. 4.5).

The center of mass of an isosceles triangle is two-thirds of the way along its height, measured from the apex towards the base of different length. (An isosceles triangle is two right-angled triangles put together. We showed above that the center of mass of a right-angled triangle is at two-thirds of the lengths of its two shorter sides.) Measured from the $y$-axis, the axis of the thin triangle removed on the left makes angle $\theta - \frac{\varphi}{2}$, and the axis of the one added on the right makes angle $\theta + \frac{\varphi}{2}$. So the height of the center of mass of the thin triangle changes by

$$\begin{aligned} \Delta y &= \frac{2}{3}R\left[\cos\!\left(\theta + \frac{\varphi}{2}\right) - \cos\!\left(\theta - \frac{\varphi}{2}\right)\right] \\ &= \frac{2}{3}R \cdot \left(-2\sin\theta\sin\frac{\varphi}{2}\right) \approx -\frac{2}{3}R\sin\theta\,\varphi. \end{aligned}$$

The second step uses the identity $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$; the last step uses $\sin x \approx x$ for small $x$.

The area of the thin triangle is $\frac{1}{2}R^2\sin\varphi \approx \frac{1}{2}R^2\varphi$ (half the product of two sides times the sine of the angle between them). So its mass is $\sigma\frac{1}{2}R^2\varphi$, and the change in the potential energy of the whole sector caused by the rotation is

$$\Delta U = \sigma\frac{1}{2}R^2\varphi \cdot g \cdot \left(-\frac{2}{3}R\sin\theta\,\varphi\right) = -\frac{1}{3}\sigma R^3 g\sin\theta\,\varphi^2.$$

We can find $\Delta U$ in a second way: from the change in the height of the center of mass of the whole sector. This works because the gravitational potential energy of an extended body in a uniform field equals that of a point mass, equal to the total mass of the body, placed at its center of mass. (Proof: the total potential energy is $\int g r_y\,\mathrm{d}m$, where $r_y$ is the vertical coordinate of the mass element $\mathrm{d}m$, and the integral runs over the whole body. Since $g$ is uniform, $\int g r_y\,\mathrm{d}m = g\int r_y\,\mathrm{d}m = Mg y_{\mathrm{CM}}$, where $M$ is the total mass. The last step is the definition of the center of mass.) When the sector turns by $\varphi$ about the origin, its center of mass moves from height $y_{\mathrm{CM}}$ to height $y_{\mathrm{CM}}\cos\varphi$. So

$$\Delta U = -Mg y_{\mathrm{CM}}(1 - \cos\varphi),$$

where $M$ is the total mass of the sector. Substitute $M = \theta\sigma R^2$ (the area of the sector is $\frac{1}{2}(2\theta)R^2$) and use the small-angle approximation $\cos x \approx 1 - \frac{x^2}{2}$:

$$\Delta U = -\frac{1}{2}\theta\sigma R^2 g y_{\mathrm{CM}}\varphi^2.$$

Setting the two expressions for $\Delta U$ equal,

$$y_{\mathrm{CM}} = \frac{2R\sin\theta}{3\theta}.$$

Putting $\theta = \frac{\pi}{2}$ gives back $y_{\mathrm{CM}} = \frac{4R}{3\pi}$ for a semi-circle.

### 4.5.4 Scaling Arguments

[CP1 4.3.1]

If we scale every length of an object by a factor $k$, the distance between any two corresponding points is also scaled by $k$. We can use this to find the center of mass of suitable objects without any integration. Take the right-angled triangle again.

**Figure 4.6**

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FIGURE 4.6. Triangle of base length $2l$.

Let the center of mass of a right-angled triangle with angle of inclination $\theta$ and base $l$ be a horizontal distance $x$ from its vertical side (its height). Then, by scaling, for the similar triangle with base $2l$ this distance is $2x$ (the hollow white circle marked "CM" in Fig. 4.6). This larger triangle is also made of 4 copies of the original triangle, as the diagram shows. So we can replace each small triangle by a point mass at its own center of mass (the black dots). Their exact heights do not matter here, since we only want the horizontal coordinate. Now we compute the horizontal coordinate $X_{\mathrm{CM}}$ of the large triangle's center of mass in two ways, measuring from its left tip:

- As the average of the four small triangles (equal masses), whose centers are at $l-x$, $l+x$, $2l-x$ and $2l-x$ (the middle triangle is upside down).
- From scaling: $2x$ from the vertical side, which is at $2l$, that is, at $2l-2x$.

This gives an equation for $x$:

$$X_{\mathrm{CM}} = \frac{l-x}{4} + \frac{l+x}{4} + \frac{2l-x}{4} + \frac{2l-x}{4} = 2l - 2x.$$

The left side is $\frac{6l-2x}{4}$, so $6l - 2x = 8l - 8x$, and

$$x = \frac{l}{3}.$$

So for the triangle with base $l$, the $x$-coordinate of the center of mass (from the left tip) is

$$x_{\mathrm{CM}} = l - x = \frac{2l}{3}.$$

A similar argument works for the $y$-coordinate.

### 4.5.5 Velocity of the Center of Mass and the Center-of-Mass Frame

[CP1 6.5, 6.5.2]

The center-of-mass frame will be useful in many later notes, for example for collisions (Note 6). It relies on Galilean transformations between inertial frames, which are covered in Note 3.

As stated before, the position vector $\vec{\boldsymbol{R}}$ of the center of mass of $N$ discrete particles, or of a continuous mass distribution, is

$$\vec{\boldsymbol{R}} = \frac{\sum_{i=1}^{N} m_i \vec{\boldsymbol{r}}_i}{\sum_{i=1}^{N} m_i} = \frac{\int \vec{\boldsymbol{r}}\,\mathrm{d}m}{\int \mathrm{d}m}.$$

Differentiating with respect to time, the velocity of the center of mass is

$$\vec{\boldsymbol{v}}_{\mathrm{CM}} = \frac{\sum_{i=1}^{N} m_i \vec{\boldsymbol{v}}_i}{\sum_{i=1}^{N} m_i} = \frac{\int \vec{\boldsymbol{v}}\,\mathrm{d}m}{\int \mathrm{d}m}.$$

The numerator is the total momentum. So if there is no net external force, the total momentum is conserved (Section 4.4), and the center of mass moves at a constant velocity $\vec{\boldsymbol{v}}_{\mathrm{CM}}$.

***The center-of-mass frame.*** Now look at the system from a frame attached to its center of mass, called the *center-of-mass frame*. Take a discrete system (the continuous case is a simple extension). Let $\vec{\boldsymbol{u}}_i$ be the velocity of the $i$th particle in the lab frame and $\vec{\boldsymbol{u}}_i'$ its velocity in the center-of-mass frame. By the Galilean velocity transformation (Note 3),

$$\vec{\boldsymbol{u}}_i' = \vec{\boldsymbol{u}}_i - \vec{\boldsymbol{v}}_{\mathrm{CM}}.$$

The total momentum in the center-of-mass frame is then

$$\begin{aligned} \vec{\boldsymbol{p}}' &= \sum_{i=1}^{N} m_i \vec{\boldsymbol{u}}_i' = \sum_{i=1}^{N} m_i \left(\vec{\boldsymbol{u}}_i - \vec{\boldsymbol{v}}_{\mathrm{CM}}\right) \\ &= M\vec{\boldsymbol{v}}_{\mathrm{CM}} - M\vec{\boldsymbol{v}}_{\mathrm{CM}}, \end{aligned}$$

where $M$ is the total mass. (The first sum is the lab-frame momentum $M\vec{\boldsymbol{v}}_{\mathrm{CM}}$ by Eq. (4.7).) So

$$\vec{\boldsymbol{p}}' = 0. \tag{4.12}$$

The total momentum of a system in its own center-of-mass frame is zero.

## 4.6 EQUATIONS OF MOTION IN DIFFERENT COORDINATES

[CP1 4.4]

The equations $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ are vector equations. To use them, we usually write them as scalar equations (components) and then solve the differential equations that result. How we do this depends on the coordinate system.

### 4.6.1 Cartesian Coordinate System

[CP1 4.4.1]

In Cartesian coordinates, the net force and the position vector are

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= F_x \hat{\boldsymbol{\imath}} + F_y \hat{\boldsymbol{\jmath}} + F_z \hat{\boldsymbol{k}}, \\ \vec{\boldsymbol{r}} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}. \end{aligned}$$

The acceleration is

$$\vec{\boldsymbol{a}} = \frac{\mathrm{d}^2\vec{\boldsymbol{r}}}{\mathrm{d}t^2}.$$

The rate of change of a vector $\vec{\boldsymbol{A}} = A\hat{\boldsymbol{A}}$ depends on the change of both its magnitude and its direction:

$$\frac{\mathrm{d}(A\hat{\boldsymbol{A}})}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\hat{\boldsymbol{A}} + A\frac{\mathrm{d}\hat{\boldsymbol{A}}}{\mathrm{d}t}.$$

The unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are fixed, so only the components change, and

$$\vec{\boldsymbol{a}} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\hat{\boldsymbol{\jmath}} + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\hat{\boldsymbol{k}}.$$

Matching the components of $\sum\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ gives

$$F_x = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \qquad F_y = m\frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \qquad F_z = m\frac{\mathrm{d}^2z}{\mathrm{d}t^2}.$$

### 4.6.2 Polar Coordinate System

[CP1 4.4.2]

In two-dimensional polar coordinates, the unit vectors are $\hat{\boldsymbol{r}}$, which points from the origin to the point of interest, and $\hat{\boldsymbol{\theta}}$, which is perpendicular to $\hat{\boldsymbol{r}}$ (tangential, in the direction of increasing $\theta$).

**Figure 4.7**
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**Figure 4.7.** Polar coordinates.

The net force and the position vector are

$$\sum\vec{\boldsymbol{F}} = F_r\hat{\boldsymbol{r}} + F_\theta\hat{\boldsymbol{\theta}}, \qquad \vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}}.$$

The acceleration is again

$$\vec{\boldsymbol{a}} = \frac{\mathrm{d}^2\vec{\boldsymbol{r}}}{\mathrm{d}t^2},$$

but now the unit vectors change as the point moves, and we must include their change when we differentiate the position vector. Note 3 derives the result in terms of the unit vectors at that instant:

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}.$$

Matching components of $\sum\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ gives

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right), \tag{4.13}$$

$$F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right). \tag{4.14}$$

## 4.7 TYPICAL FORCES IN MECHANICS

[CP1 4.5]

We can now find how a system evolves once we know the forces on it. So we turn to working out the forces themselves.

There are four fundamental interactions in nature: gravitational, electromagnetic, strong nuclear and weak nuclear. All *mechanical* forces, which need direct contact to act, are in fact electromagnetic. These include the normal force, friction and the spring force. When you push on a door, the electrons in your fingertips are pressed against those in the door; they repel, and this is the normal force. As another example, the tension in a string comes from the attraction between the atoms of the string, which stops the string from coming apart when it is stretched. Of the forces in this section, only the gravitational force is fundamental.

### 4.7.1 Normal Force

[CP1 4.5.1]

The normal force, as its name says, acts *perpendicular* (normal) to the surface between two objects. When we stand on the ground, the normal force from the ground opposes the gravitational force and stops us from accelerating towards the center of the Earth. The normal force is a contact force: it acts at the point of contact between the objects. This matters a lot when we study torques (Note 5). Finally, an object's *apparent weight* is the normal force between the object and an imaginary weighing scale (the scale pushes on the object and the object pushes on the scale with the same size of force).

### 4.7.2 Friction

[CP1 4.5.2]

Friction is a force that resists relative motion between two surfaces. It comes from several causes, such as surfaces sticking together, surfaces deforming and surfaces being rough. We must tell apart static friction and kinetic friction.

*Static friction* acts when the two surfaces do not move relative to each other. It obeys

$$|f_s| \le \mu_s N,$$

where $|f_s|$ is the size of the static friction, $\mu_s$ is the coefficient of static friction and $N$ is the normal force between the surfaces. Static friction points so as to oppose the motion that would happen without it. If the force needed to prevent sliding would be larger than the upper limit $\mu_s N$, the object starts to move.

*Kinetic friction*, on the other hand, opposes the motion of surfaces that are already moving relative to each other. It is constant, with size

$$f_v = \mu_v N.$$

In most cases $0 < \mu < 1$ and $\mu_v < \mu_s$. There are exceptions, such as silicone rubber surfaces, whose coefficients of friction are often much larger than 1. Finally, it is important to understand that kinetic friction always acts opposite to the motion, so it is a *non-conservative* and dissipative force (Note 6). The energy is usually turned into heat and sound, which go into the surroundings.

### 4.7.3 Spring Force

[CP1 4.5.3]

When a spring, or any elastic object, is compressed or stretched, it tends to spring back to its natural state. By *Hooke's law*, this restoring force is proportional to the extension or compression $x$ of the spring relative to its relaxed length:

$$F = kx.$$

The force points so that the spring tends to return to its relaxed length.

Let us step aside and find the *effective spring constant* of massless springs with equal relaxed lengths, joined in parallel and in series. Take a system of two springs. Let the displacements of the object, the first spring and the second spring from their equilibrium positions be $x$, $x_1$ and $x_2$.

***Parallel configuration.*** Figure 4.8 shows that in parallel, the springs and the object all move by the same amount:

$$x = x_1 = x_2.$$

**Figure 4.8**

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FIGURE 4.8. Springs connected in parallel.

The total force on the object is

$$F = k_1 x_1 + k_2 x_2 = (k_1 + k_2)x.$$

The effective spring constant $k_{\text{eff}}$ is the one for which

$$F = k_{\text{eff}} x \quad \Longrightarrow \quad k_{\text{eff}} = k_1 + k_2.$$

Using this formula again and again on pairs of springs, the effective spring constant for $n$ springs in parallel is

$$k_{\text{eff}} = \sum_{i=1}^{n} k_i.$$

***Series configuration.*** For springs in series (Fig. 4.9), each spring must exert the same force. Otherwise there would be a net force on a massless spring, which would give it an infinite acceleration. So

$$F = k_{\text{eff}} x = k_1 x_1 = k_2 x_2.$$

Also, the extensions of the two springs add up to the displacement of the object:

$$x_1 + x_2 = x.$$

To solve, multiply $k_1 x_1 = k_{\text{eff}} x$ by $k_2$, and $k_2 x_2 = k_{\text{eff}} x$ by $k_1$, then add:

$$\begin{aligned} k_1 k_2 x_1 &= k_2 k_{\text{eff}} x \\ k_1 k_2 x_2 &= k_1 k_{\text{eff}} x \\ k_1 k_2 (x_1 + x_2) &= (k_1 + k_2) k_{\text{eff}} x \end{aligned}$$

$$\Longrightarrow \quad k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2} \quad \text{or} \quad \frac{1}{k_{\text{eff}}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

Using this again and again, the effective spring constant for $n$ springs in series is

$$\frac{1}{k_{\text{eff}}} = \sum_{i=1}^{n} \frac{1}{k_i}.$$

**Figure 4.9**

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FIGURE 4.9. Springs connected in series.

### 4.7.4 Tension

[CP1 4.5.4]

Tension is the pulling force that each part of a string exerts on the parts next to it, and at the ends of the string. You can picture a string or rope as a long spring, with the tension as its restoring force. But this picture fails when the string is pushed together: it goes slack instead of acting like a compressed spring. The tension at any point of a string always points along the string at that point (along its local direction), because a string cannot resist forces perpendicular to it.

In most problems, strings and ropes are taken to be massless, so the net force on them must be zero. This is exactly what makes the tension in a frictionless, massless string of any shape uniform (the same everywhere), as we see later in this note (Section 4.8.3). Another proof is given in Note 7.

### 4.7.5 Gravitational Force

[CP1 4.5.5]

The gravitational force is an attraction between all particles that have mass. In its general Newtonian form, the size of the gravitational force between two point masses is proportional to the product of their masses and inversely proportional to the square of the distance between them (Note 8).

In the rest of this note, the only massive body we consider is the Earth, and all differences in height are tiny compared with the Earth's radius. So we can take the gravitational force, or *weight*, on an object of mass $m$ to be $mg$, where $g$ is the gravitational field strength (a constant).

For an object that is not tiny, the *center of gravity* is the point where the whole force of gravity seems to act. It is the same as the center of mass when the gravitational field strength is uniform over the whole object (try to prove this).

## 4.8 SOLVING PROBLEMS

[CP1 4.6]

### 4.8.1 Free-Body Diagrams

[CP1 4.6.1]

Before looking at the common types of problems, we need a key tool: the *free-body diagram*. Draw it like this:

(1) Isolate the system you are studying.

(2) Find *all* external forces on the system and draw them as vectors at the right points of the system (for example, the gravitational force should pass through the center of gravity). External forces are forces on the system due to things outside the system.

The general method for solving a mechanics problem is then:

(1) Choose several systems and draw their free-body diagrams. A useful tip for choosing a system: look at the forces you need to find. If they do not include internal forces (friction and normal forces between surfaces, and sometimes tension), treat the objects together as one system. If you need the friction or normal force between two objects, separate them.

(2) Write the $\sum \vec{F} = m\vec{a}$ equations for all the systems you chose.

(3) If needed, find relations between the variables (*constraints* that must be obeyed). They give extra equations, so that you have enough equations for the unknowns. Sometimes the constraints can instead be built into the coordinates that describe each system. (Such constraints are called *holonomic*: they depend only on the positions of the objects. CP1 Chapter 11 explores this further.)

(4) Solve!

Let us apply this method to some problems.

### 4.8.2 No Constraints

[CP1 4.6.2]

**Example.** [CP1 4.6.2, p.154] A man of mass $m$ is in a lift that accelerates upwards with acceleration $a$. Find the normal force on the man from the floor of the lift (his apparent weight).

***Solution*** We want the normal force on the man, so we isolate the man and draw his free-body diagram (Fig. 4.10). In the figure, the normal force is drawn slightly away from its exact position so that it can be seen clearly. The acceleration $a$ is drawn at the side only as a reminder that the man accelerates upwards; this is not really needed.

**Figure 4.10**

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**Figure 4.10.** A man in an elevator.

Two forces act on him: the normal force $N$ upwards and his weight $mg$ downwards. Taking upwards as positive, Newton's second law gives

$$N - mg = ma \quad \Longrightarrow \quad N = m(a + g).$$

**Example.** [CP1 4.6.2, p.154] A force $F$ is applied to the left end of a row of $n$ identical blocks of mass $m$, connected by strings (Fig. 4.11). The coefficient of kinetic friction between the blocks and the ground is $\mu$. All strings stay taut while the blocks move. Find the tension $T_i$ in the string between the $i$th and $(i+1)$th blocks as the blocks accelerate.

**Figure 4.11**

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**Figure 4.11.** Connected blocks.

***Solution*** Let $f$ be the friction force on one block. First find the acceleration by treating all the blocks as one system. Since the strings stay taut, all blocks have the same acceleration $a$. The strings are internal to this system, so their tensions do not appear:

$$F - nf = nma.$$

Next, a convenient system is the $(i+1)$th block up to the $n$th block, because $T_i$ is the only tension acting on it. (Choosing a good system needs some feel for the problem, which gets better with practice.) Its free-body diagram is in Fig. 4.12.

**Figure 4.12**

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**FIGURE 4.12.** The last $(n-i)$ blocks.

This system has $n-i$ blocks, so

$$T_i - (n-i)f = (n-i)ma.$$

From the first equation, $f + ma = \frac{F}{n}$. So

$$T_i = (n-i)(f+ma) = \frac{(n-i)F}{n}.$$

We never needed to find $f$! So if we did this experiment on the Moon, for example, where $g$ and hence $f$ are different, we would get exactly the same result.

An equally convenient system is the first $i$ blocks. It feels the force $F$, the tension $-T_i$ (pulling backwards), the friction $-if$ and the weight $img$. Horizontally,

$$\begin{aligned} F - T_i - if &= ima \\ T_i = F - i(f+ma) &= F - \frac{iF}{n} = \frac{(n-i)F}{n}. \end{aligned}$$

***Slinkies.*** A simple but fascinating system is the toy called a *slinky*, which is basically a spring with mass. The main difficulty is that the spring force is no longer the same all along the slinky, because each part now has mass. The spring is not stretched evenly, so its density changes along its length, even if it was uniform to begin with. Because the spring force varies, the standard careful method is to study infinitesimal pieces of the slinky, each of which is a small spring with mass. But we will see that some clever tricks can avoid this.

**EXAMPLE.** [CP1 4.6.2, p.156] A slinky has mass $m$, spring constant $k$ and relaxed length $l$. (The relaxed length is its length when the tensions at both ends are zero, for example when it lies on a horizontal table.) It is hung from a ceiling. After it reaches equilibrium, find the total extension of the slinky and the position of its center of mass.

***Solution*** *Extension.* Divide the original, unstretched slinky into a huge number of equal tiny pieces, each of length $dx$. Each piece has the same spring constant $k'$ (we find it later). We can find the extension without any long calculation. At equilibrium, the spring force grows linearly from 0 at the bottom of the slinky to $mg$ at the ceiling, because each piece must hold up the weight of all the pieces below it. (Here "linearly" is with respect to the piece's position counted along the *original* slinky from the bottom, not the stretched length from the bottom.) So each piece stretches by a different amount. The key point is that the extension of each tiny piece is directly proportional to the tension at its ends. Since the tension varies linearly, the total extension of the slinky equals that of a spring with the same constant $k$ and a uniform tension equal to the *average* tension, $\frac{mg}{2}$. That is the same as hanging a block of weight $\frac{mg}{2}$ on a massless spring. So the extension of the slinky is simply

$$\Delta l = \frac{mg}{2k}.$$

*Center of mass.* Now we are stuck, because this trick does not work for the second part: the whole point is to find the center of mass of a spring whose mass density is not uniform. We can guess the answer's form. The tension is larger near the top of the slinky, so the top parts stretch more than the bottom. So the density of the slinky falls with height, and the center of mass lies *below* the geometric center. We now use the careful method. Put the origin at the ceiling and take the $x$-direction as positive downwards.

**Figure 4.13**

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**FIGURE 4.13.** Infinitesimal section of slinky.

First imagine switching off gravity. The slinky then runs from $x=0$ to $x=l$. Isolate a tiny piece between $x$ and $x+dx$.

A spring of constant $k$ and length $l$ can be seen as $N$ springs, each of length $\frac{l}{N}$, joined in series. By the series formula $\frac{1}{k} = \sum \frac{1}{k_N} = \frac{N}{k_N}$, the spring constant of each small spring is

$$k_N = Nk.$$

In other words, the length $l'$ of a part of the spring times its spring constant $k'$ always equals $kl$:

$$k'l' = kl.$$

For the tiny piece of length $dx$, the spring constant is therefore

$$k' = \frac{kl}{dx}.$$

Do not worry about the infinitesimal in the denominator for now.

Now switch gravity on and let the system settle. The tiny piece ends up between $x + \varepsilon(x)$ and $x + dx + \varepsilon(x+dx)$, where $\varepsilon(x)$ is how far down the point that was originally at $x$ has moved (Fig. 4.13). The extension of this piece is $\varepsilon(x+dx) - \varepsilon(x)$.

As a function of $x$ (remember that $x$ labels points of the *original* spring), the tension must hold up the weight of the part below the piece, that is, of the part with larger $x$. That part has mass $m\left(1 - \frac{x}{l}\right)$, so

$$T = mg\left(1 - \frac{x}{l}\right).$$

This tension on the ends of the piece stretches it according to Hooke's law:

$$\begin{aligned} T &= k'\left[\varepsilon(x+dx) - \varepsilon(x)\right] \\ &= kl\,\frac{\varepsilon(x+dx) - \varepsilon(x)}{dx} = kl\frac{d\varepsilon}{dx}. \end{aligned}$$

Substitute $T$, separate variables and integrate:

$$\int_0^{\varepsilon(x)} d\varepsilon = \int_0^x \frac{mg}{kl}\left(1 - \frac{x}{l}\right) dx,$$

where $\varepsilon(0) = 0$ because the point at the ceiling is fixed. So the total extension of the part between the ceiling and $x$ is

$$\varepsilon(x) = \frac{mg}{kl}\left(x - \frac{x^2}{2l}\right).$$

Putting $x = l$ checks that the total extension of the slinky is indeed $\varepsilon(l) = \frac{mg}{2k}$.

The center of mass of each tiny piece is at either of its ends (the difference is second order and can be ignored). The final position of the top end of the piece that was originally between $x$ and $x + \mathrm{d}x$ is

$$x' = x + \varepsilon(x) = \left(\frac{mg}{kl} + 1\right)x - \frac{mgx^2}{2kl^2}.$$

Although this piece is stretched, its mass is still $\rho\,\mathrm{d}x$, where $\rho = \frac{m}{l}$ is the original density of the slinky, because it still contains the same material. To find the center of mass we just integrate

$$\int_0^l \rho x'\,\mathrm{d}x\,.$$

Be careful not to confuse $\mathrm{d}x'$ with $\mathrm{d}x$: the mass of the piece is $\rho\,\mathrm{d}x$, not its new density times $\mathrm{d}x'$. (You could work with the new density and $\mathrm{d}x'$, but it is an unnecessary extra step.)

$$\begin{aligned}\int_0^l \rho x'\,\mathrm{d}x &= \rho\int_0^l \left[\left(\frac{mg}{kl} + 1\right)x - \frac{mgx^2}{2kl^2}\right]\mathrm{d}x \\ &= \rho\left[\left(\frac{mg}{kl} + 1\right)\frac{l^2}{2} - \frac{mgl}{6k}\right] = \frac{ml}{2} + \frac{m^2g}{3k}.\end{aligned}$$

Dividing by $m$, the $x$-coordinate of the center of mass is

$$x_{\mathrm{CM}} = \frac{l}{2} + \frac{mg}{3k},$$

which is $\frac{mg}{3k}$ below the center of the unstretched slinky.

*A neater method: scaling.* Having done the careful analysis, here is an elegant alternative based on scaling arguments.

The possible parameters are $m$, $g$, $k$ and $l$. By dimensional analysis, a possible form is

$$x_{\mathrm{CM}} = \alpha l + \beta\frac{mg}{k},$$

where $\alpha$ and $\beta$ are dimensionless constants. Now, $\frac{mg}{kl}$ is also dimensionless, but we do not allow arbitrary functions of $\frac{mg}{kl}$ in our guess. The reason: the equations of the system (Hooke's law, and the tension as a function of $x$) are linear, and $l$ and $\frac{mg}{k}$ should be independent of each other. (We could have a slinky with $l = 0$ or $m = 0$, for example, and the same formula should still work. If $l$ and $\frac{mg}{k}$ were mixed together, these limiting cases would give nonsense.) So it is sensible to guess a solution that is linear in $l$ and $\frac{mg}{k}$.

We find $\alpha$ by letting $k \to \infty$: the slinky becomes so stiff that it hardly stretches. Then

$$x_{\mathrm{CM}} = \frac{l}{2} \quad\Longrightarrow\quad \alpha = \frac{1}{2}.$$

Define $\Delta x$ as the extra distance of $x_{\mathrm{CM}}$ beyond the center of mass of the relaxed slinky:

$$\Delta x = x_{\mathrm{CM}} - \frac{l}{2} = \beta\frac{mg}{k}.$$

Notice that

$$\Delta x \propto \frac{m}{k}.$$

So if we hang a slinky of the same linear mass density but length $2l$, the shift of its center of mass from the relaxed position should be $4\Delta x$: its mass is $2m$, and its spring constant is $\frac{k}{2}$ (two identical springs in series).

To get a second expression for this shift, cut the long slinky into two parts of equal mass (Fig. 4.14). They do *not* have equal lengths at equilibrium, because the top part is stretched more.

**Figure 4.14**

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**Figure 4.14.** Scaling arguments for a slinky of length $2l$.

The *bottom* part is a slinky of mass $m$ and original length $l$, with tension $mg$ at its top and 0 at its bottom. That is exactly the hanging slinky we started with, so its center of mass is $\frac{l}{2} + \Delta x$ below its top end, by assumption.

The *top* part is a bit harder. It is a slinky of mass $m$ and original length $l$, but with tensions $2mg$ and $mg$ at its ends (Fig. 4.14). This is the same as the original slinky with an extra mass $m$ hung from its bottom end. The extra weight $mg$ stretches the slinky *uniformly* by an extra $\frac{mg}{k}$, which moves its center down by an extra $\frac{mg}{2k}$. So the center of mass of the top part is at

$$\frac{l}{2} + \Delta x + \frac{mg}{2k}.$$

The top part's total length is $l + \Delta l + \frac{mg}{k}$, so the bottom part's center of mass is at

$$\begin{aligned}l + \Delta l + \frac{mg}{k} + \frac{l}{2} + \Delta x &= \frac{3}{2}l + \frac{mg}{2k} + \frac{mg}{k} + \Delta x \\ &= \frac{3}{2}l + \frac{3mg}{2k} + \Delta x.\end{aligned}$$

Remember that $\Delta l = \frac{mg}{2k}$ is the extension of the slinky under its own weight. The two parts have equal masses, so the center of mass of the whole slinky of length $2l$ is the average of the two positions:

$$\begin{aligned}x_{\mathrm{CM}} &= \frac{\frac{l}{2} + \Delta x + \frac{mg}{2k} + \frac{3}{2}l + \frac{3mg}{2k} + \Delta x}{2} \\ &= l + \Delta x + \frac{mg}{k}.\end{aligned}$$

From the scaling argument we also know that

$$x_{\mathrm{CM}} = \frac{2l}{2} + 4\Delta x = l + 4\Delta x.$$

Setting these equal,

$$\Delta x = \frac{mg}{3k} \quad\Longrightarrow\quad x_{\mathrm{CM}} = \frac{l}{2} + \frac{mg}{3k}.$$

### 4.8.3 Conservation of String

[CP1 4.6.3]

Another important class of problems is *Atwood's machines*: systems of pulleys, strings and masses. In this note we only consider massless pulleys and strings, though this is not true in general.

Besides writing the $F = ma$ equation for each mass, we must notice that the length of each string is "conserved". Since a string cannot stretch, the positions of the objects at its two ends must obey a fixed relation. This gives extra equations that relate the accelerations of the masses. The next few examples show how. Atwood's machines are usually one-dimensional, so we do not use vector notation for them. In all Atwood's machines in this note, *downwards is the positive direction*.

**EXAMPLE.** [CP1 4.6.3, p.161] Find the accelerations $a_1$ and $a_2$ of $m_1$ and $m_2$, and the tensions, in the set-up of Fig. 4.15.

**Figure 4.15**

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**FIGURE 4.15.** Atwood's machine 1.

***Solution*** First, look at the tiny piece of string at the top of the pulley. The forces on it must balance, since otherwise this massless piece would have an infinite acceleration. So

$$T_1 = T_2.$$

From now on we write $T$ for this tension. In fact, we can repeat this argument for every tiny piece of a massless string, so the tension is the same all along a continuous piece of string. So $T$ is, without doubt, the tension in the string joining $m_1$ and $m_2$.

The $F = ma$ equations for the two masses are

$$m_1 a_1 = m_1 g - T,$$
$$m_2 a_2 = m_2 g - T.$$

Finally, by conservation of string,

$$a_1 = -a_2.$$

This is because if $m_1$ moves up by some distance $d$, $m_2$ must move down by $d$ so that the length of string stays the same.

When solving such equations, it is usually quickest to multiply the $F = ma$ equations by factors chosen so that the left sides add up to zero by the conservation of string equation. Then we can solve directly for the tension. Here, multiply the first equation by $m_2$ and the second by $m_1$, and add:

$$m_1 m_2 (a_1 + a_2) = 2 m_1 m_2 g - (m_1 + m_2) T.$$

Since $a_1 + a_2 = 0$,

$$T = \frac{2 m_1 m_2}{m_1 + m_2} g.$$

Putting $T$ back into the $F = ma$ equations,

$$a_1 = \frac{m_1 - m_2}{m_1 + m_2} g,$$
$$a_2 = -a_1 = \frac{m_2 - m_1}{m_1 + m_2} g.$$

**EXAMPLE.** [CP1 4.6.3, p.162] Find all tensions and accelerations in the two-layer system of three masses in Fig. 4.16.

**Figure 4.16**

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**FIGURE 4.16.** Atwood's machine 2.

***Solution*** We already know how the tensions in different parts are related. By the argument above, the tension in the string over the bottom pulley is the same everywhere; call it $T$. The tension in the string over the top pulley must then be $2T$, so that the massless bottom pulley (together with the bottom string) has no net force on it.

Why must a massless object have zero net force, even though it can accelerate? Newton's second law for a massless object gives $a = \frac{F}{m}$. If both $F \to 0$ and $m \to 0$, then $a$ is not fixed by this formula; it depends on how $F$ and $m$ go to zero. So $a$ can be non-zero even though there is no net force. But a non-zero net force on a massless object would give an infinite acceleration.

So we now have two rules:

- the tension is the same all along a massless, frictionless string;
- the forces on every massless pulley must balance.

From now on, we write the tensions in different parts of the string directly, without mentioning these rules each time, because the relations are usually simple. If you are ever unsure why a tension has a certain value, go back to these two rules and work it out yourself.

As always, write the $F = ma$ equations for all three masses:

$$m_1 a_1 = m_1 g - 2T,$$
$$m_2 a_2 = m_2 g - T,$$
$$m_3 a_3 = m_3 g - T.$$

The conservation of string equation is less obvious here. It is

$$a_1 = -\frac{a_2 + a_3}{2}.$$

This is because the average height of $m_2$ and $m_3$ moves by the same distance as the bottom pulley, and the bottom pulley moves by the same distance as $m_1$, in the opposite direction.

If this is still not clear, split the motion of $m_2$ and $m_3$ into two parts (Fig. 4.17). The first part is due to the motion of the pulley they hang from, which accelerates at $-a_1$ by conservation of the top string. The second part is the acceleration $a_f$ of the string joining $m_2$ and $m_3$ around a pulley that is at rest (since we have already taken out the pulley's motion). So

$$a_2 = -a_1 - a_f,$$
$$a_3 = -a_1 + a_f.$$

Adding these two equations and dividing by 2,

$$a_1 = -\frac{a_2 + a_3}{2}.$$

Figure 4.17
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to the `figures/` folder

FIGURE 4.17. Different components of motion.

To solve the four equations, multiply the $F = ma$ equations by $2m_2m_3$, $m_1m_3$ and $m_1m_2$ respectively and add them. The left side becomes $m_1m_2m_3(2a_1 + a_2 + a_3) = 0$, which leaves an equation for $T$ alone. The result is

$$\begin{aligned} T &= \frac{4m_1m_2m_3}{4m_2m_3 + m_1(m_2 + m_3)}g, \\ a_1 &= \frac{m_1(m_2 + m_3) - 4m_2m_3}{4m_2m_3 + m_1(m_2 + m_3)}g, \\ a_2 &= \frac{4m_2m_3 + m_1(m_2 - 3m_3)}{4m_2m_3 + m_1(m_2 + m_3)}g, \\ a_3 &= \frac{4m_2m_3 + m_1(m_3 - 3m_2)}{4m_2m_3 + m_1(m_2 + m_3)}g. \end{aligned}$$

EXAMPLE. [CP1 4.6.3, p.164] Now add a movable pulley that spans the width of a single layer. Find all tensions and accelerations in the system of Fig. 4.18.

Figure 4.18
upload: `v1_ch04_fig4-18`.png / .pdf / .jpg
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FIGURE 4.18. Atwood's machine 3.

***Solution*** The $F = ma$ equations are

$$m_1a_1 = m_1g - T,$$
$$m_2a_2 = m_2g - 2T,$$

where the tension on $m_2$ is $2T$ so that the forces on the right pulley balance. If the right pulley moves down by $x$, then $m_1$ must move up by $2x$ to keep the length of string the same. So

$$a_1 = -2a_2.$$

To solve, multiply the $F = ma$ equations by $m_2$ and $2m_1$ and add (the left side becomes $m_1m_2(a_1 + 2a_2) = 0$):

$$\begin{aligned} T &= \frac{3m_1m_2}{4m_1 + m_2}g, \\ a_1 &= \frac{4m_1 - 2m_2}{4m_1 + m_2}g, \\ a_2 &= \frac{m_2 - 2m_1}{4m_1 + m_2}g. \end{aligned}$$

***Building constraints into coordinates.*** The derivations above give a clear picture of what happens physically. But it is often easier to get the conservation of string equation by defining the coordinate of each mass while taking the length of each string into account. Then the accelerations of the masses can be written in terms of fewer independent coordinates. While doing this, it is convenient to take the lengths of all strings and the circumferences of all pulleys to be *zero*. They are constants, so they disappear anyway when we differentiate.

EXAMPLE. [CP1 4.6.3, p.165] Three identical masses $m$ are connected as in Fig. 4.19. Find their accelerations.

Figure 4.19
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FIGURE 4.19. Atwood's machine 4.

***Solution*** Number the masses 1, 2, 3 from left to right. Let the coordinate of the first mass be $x$ (positive downwards). The string joining the first mass to its pulley has zero length (by our convention), so that pulley's coordinate is $0 - x = -x$. Let the coordinate of the third mass, and so of the pulley it is attached to, be $y$. Then the segment between the second and third pulleys has length $y - (-x) = y + x$. Finally, the segment between the top of the second pulley and the second mass has length $0 - y - (y + x) = -2y - x$, since the string over the second and third pulleys has zero length. So the coordinates of the three masses are

$$x, \qquad -x + (-2y - x) = -2x - 2y, \qquad y.$$

Hence

$$a_1 = \ddot{x}, \qquad a_2 = -2\ddot{x} - 2\ddot{y}, \qquad a_3 = \ddot{y}.$$

Clearly, the conservation of string equation is

$$2a_1 + a_2 + 2a_3 = 0.$$

Let the tension on the second mass be $T$. Then the tensions on the first and third masses are both $2T$. $F = ma$ gives

$$\begin{aligned} ma_1 &= mg - 2T, \\ ma_2 &= mg - T, \\ ma_3 &= mg - 2T. \end{aligned}$$

Put these into the string equation: $2\left(g - \frac{2T}{m}\right) + \left(g - \frac{T}{m}\right) + 2\left(g - \frac{2T}{m}\right) = 5g - \frac{9T}{m} = 0$. Solving,

$$T = \frac{5}{9}mg, \quad a_1 = -\frac{1}{9}g, \quad a_2 = \frac{4}{9}g, \quad a_3 = -\frac{1}{9}g.$$

## 4.8.4 Remaining on an Inclined Plane

[CP1 4.6.4]

In some problems, an object must stay on a surface such as an inclined plane. Then the acceleration of the object relative to the surface must obey a certain relation. Consider this problem.

EXAMPLE. [CP1 4.6.4, p.167] A block of mass $m$ lies on a frictionless plane of mass $M$ and angle of inclination $\theta$. You keep the horizontal acceleration of the plane equal to $A$. Find the acceleration of the block and the normal force on the block from the plane.

**Figure 4.20**

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to the `figures/` folder

FIGURE 4.20. Block on an accelerating inclined plane.

***Solution*** As usual, write the $F = ma$ equations in the horizontal and vertical directions:

$$\begin{aligned} N\sin\theta &= ma_x, \\ N\cos\theta - mg &= ma_y. \end{aligned}$$

There are three unknowns ($N$, $a_x$ and $a_y$) but only two equations, so we need one more. It comes from a careful look at the geometry: the block must stay in contact with the plane. This constraint is

$$\frac{a_y}{a_x - A} = -\tan\theta.$$

It says that, relative to the plane, the block's vertical acceleration must be $\tan\theta$ times its horizontal acceleration, so that the block moves along the surface. ($a_x - A$ is the block's horizontal acceleration relative to the plane; the plane has no vertical acceleration.) The minus sign is because if the block moves up relative to the plane, it must move left relative to the plane.

Solving these three equations (put $a_x = \frac{N\sin\theta}{m}$ and $a_y = \frac{N\cos\theta}{m} - g$ into the constraint) gives

$$\begin{aligned} a_x &= g\sin\theta\cos\theta + A\sin^2\theta, \\ a_y &= A\sin\theta\cos\theta - g\sin^2\theta, \\ N &= mg\cos\theta + mA\sin\theta. \end{aligned}$$

Another approach is to define two independent coordinates. Let the top-left tip of the plane be at $(x, 0)$, and let $s$ be the distance of the block from the top-left tip, measured along the slope. Then the block is at $(x + s\cos\theta, -s\sin\theta)$. So

$$\begin{aligned} a_x &= \ddot{x} + \ddot{s}\cos\theta = A + \ddot{s}\cos\theta, \\ a_y &= -\ddot{s}\sin\theta, \end{aligned}$$

and the $F = ma$ equations become

$$\begin{aligned} N\sin\theta &= m(A + \ddot{s}\cos\theta), \\ N\cos\theta - mg &= -m\ddot{s}\sin\theta. \end{aligned}$$

Solving these gives the same result as before, because the condition that the block stays on the plane is quietly built into the choice of coordinates. To show this:

$$\frac{a_y}{a_x - A} = -\frac{\ddot{s}\sin\theta}{\ddot{s}\cos\theta + \ddot{x} - A} = -\tan\theta,$$

since $\ddot{x} = A$.

## 4.8.5 Polar Coordinates

[CP1 4.6.5]

Finally, we practise solving some systems in polar coordinates. An important constraint here is circular motion. Note 3 shows that for an object to stay on a circle of constant radius $r$, it must have an instantaneous centripetal acceleration, pointing radially inwards, equal to

$$a_r = -\frac{v^2}{r},$$

where $v$ is its instantaneous tangential speed. So there must be a net centripetal force on the object, which obeys

$$\sum \vec{\boldsymbol{F}}_r = -\frac{mv^2}{r}\hat{\boldsymbol{r}} = -mr\dot{\theta}^2\hat{\boldsymbol{r}},$$

by Eq. (4.13), since $\ddot{r} = 0$.

EXAMPLE. [CP1 4.6.5, p.169] A conical pendulum of length $l$ moves in uniform circular motion at a constant height, with angular velocity $\omega$ (Fig. 4.21). Find the range of $\omega$ for which the pendulum can keep up this motion with $\theta > 0$. What happens if $\omega$ is smaller than the lower limit of this range?

**Figure 4.21**

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**Figure 4.21.** Conical pendulum.

***Solution*** For the pendulum bob to stay at the same height, the net vertical force must be zero:

$$T\cos\theta = mg.$$

The bob moves in uniform circular motion with radius $l\sin\theta$. So the horizontal (radial) part of the tension must supply the required centripetal force:

$$T\sin\theta = ml\sin\theta\,\omega^2.$$

Since we are looking at positions with $\theta > 0$, we can cancel $\sin\theta$:

$$T = ml\omega^2.$$

Dividing the first equation by this,

$$\cos\theta = \frac{mg}{ml\omega^2} = \frac{g}{l\omega^2}.$$

Since $|\cos\theta| \leq 1$, the motion is possible only if

$$\omega \geq \sqrt{\frac{g}{l}}.$$

When $\omega < \sqrt{\frac{g}{l}}$, the mathematics breaks down at the step where we cancelled $\sin\theta$, because in this case $\theta = 0$ (the pendulum cannot keep up circular motion; it just hangs straight down). So $\theta$ as a function of $\omega$ is in fact

$$\theta = \begin{cases} \cos^{-1}\dfrac{g}{l\omega^2} & \text{for } \omega \geq \sqrt{\dfrac{g}{l}}, \\ 0 & \text{for } \omega < \sqrt{\dfrac{g}{l}}. \end{cases}$$

Now a more general use of $F = ma$ in polar coordinates.

**Example.** [CP1 4.6.5, p.170] Two masses are connected by a string that passes through a hole in a horizontal table (Fig. 4.22). Mass $m$ lies on the table. At this moment it is a distance $r$ from the hole, with radial velocity $v_r$ (positive outwards) and angular velocity $\omega$. Find the instantaneous acceleration of the mass $M$ below the table and the instantaneous angular acceleration of mass $m$.

**Figure 4.22**

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**Figure 4.22.** Two masses.

***Solution*** Let the tension in the string be $T$. The string pulls $m$ towards the hole, that is, in the $-\hat{\boldsymbol{r}}$ direction. Newton's laws for $m$ in polar coordinates in the plane of the table, Eqs. (4.13) and (4.14), give

$$\begin{aligned} -T &= m\ddot{r} - mr\dot{\theta}^2, \\ 0 &= mr\ddot{\theta} + 2m\dot{r}\dot{\theta}. \end{aligned}$$

At this instant $\dot{r} = v_r$ and $\dot{\theta} = \omega$, so

$$\begin{aligned} -T &= m\ddot{r} - mr\omega^2, \\ 0 &= mr\ddot{\theta} + 2mv_r\omega. \end{aligned}$$

For mass $M$, with $z$ its downward coordinate,

$$Mg - T = M\ddot{z}.$$

Conservation of string requires $\ddot{r} = -\ddot{z}$: if $M$ goes down, $m$ moves towards the hole. Then the first equation gives $T = mr\omega^2 + m\ddot{z}$. Putting this into the equation for $M$ and solving,

$$\begin{aligned} \ddot{z} &= \frac{Mg - mr\omega^2}{m + M}, \\ \ddot{\theta} &= -\frac{2v_r\omega}{r}. \end{aligned}$$

All these quantities are instantaneous. The second result shows that if the mass on the table moves radially outwards, its angular velocity decreases a moment later, and vice versa. This is expected, because the angular momentum of this system about the hole must be conserved (Note 5).

### 4.8.6 Rigid Body Constraint

[CP1 4.6.6]

Another common constraint is the *rigid body* condition: the particles must keep fixed distances from one another. A typical set-up is discrete particles joined by massless, rigid rods. (A continuous rigid body is better described by other methods; see Note 5.)

A rigid rod usually feels forces only at its two ends. The force from such a rod (its "tension") can then only point along the rod, so that the forces and torques on the rod balance. Here is why: the two equal and opposite forces at the ends must cancel each other, and their lines of action must coincide, so they must lie along the rod. Otherwise, taking moments about one end would give a non-zero net torque (Note 5). This fact, together with the kinematics of rigid body motion in Note 3, is the key to such problems.

**EXAMPLE.** [CP1 4.6.6, p.171] Three masses $m_1$, $m_2$ and $m_3$ are connected by massless, rigid rods of lengths $l_1$, $l_2$ and $l_3$ to a common massless center O. The joint at O is fixed, so that adjacent rods make constant angles $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_3 > 0$. The set-up lies on a frictionless, horizontal table. The masses are given velocities $v_1$, $v_2$ and $v_3$, perpendicular to their rods and in the clockwise direction. Find the instantaneous acceleration of each mass.

Figure 4.23

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**FIGURE 4.23.** Three masses and rigid rods.

***Solution*** *Step 1: O is at rest.* Define unit vectors $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ along the rods, pointing out from the center O (which is also the origin). Let the velocity of O be $\vec{\boldsymbol{v}}_O$. The rigid body condition says that the two ends of a rod have no relative velocity *along* the rod (otherwise the rod would change length). For all three rods,

$$(\vec{\boldsymbol{v}}_O - \vec{\boldsymbol{v}}_i) \cdot \hat{\boldsymbol{e}}_i = 0$$

for all $1 \leq i \leq 3$, where $\vec{\boldsymbol{v}}_i$ is the velocity of the $i$th mass. Since each $\vec{\boldsymbol{v}}_i$ is perpendicular to its rod, $\vec{\boldsymbol{v}}_i \cdot \hat{\boldsymbol{e}}_i = 0$, so

$$\vec{\boldsymbol{v}}_O \cdot \hat{\boldsymbol{e}}_i = 0$$

for all $i$. The $\hat{\boldsymbol{e}}_i$ point in different directions, so the only way to satisfy this is $\vec{\boldsymbol{v}}_O = 0$. (More precisely: any two of the $\hat{\boldsymbol{e}}_i$ form a basis for the two-dimensional plane of the table. Any vector in that plane is a combination of them, so a vector whose dot products with both of them are zero must be the zero vector.)

*Step 2: the tensions.* Let the forces exerted by the rods on the massless center O be $\vec{\boldsymbol{T}}_1$, $\vec{\boldsymbol{T}}_2$ and $\vec{\boldsymbol{T}}_3$. The forces on O must balance, or it would have an infinite acceleration:

$$\vec{\boldsymbol{T}}_1 + \vec{\boldsymbol{T}}_2 + \vec{\boldsymbol{T}}_3 = 0.$$

So the three tension vectors form a closed force triangle (Fig. 4.24). The angles between them are known, because each points along its rod.

Figure 4.24

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**FIGURE 4.24.** Force triangle.

By the sine rule,

$$\frac{T_1}{\sin\theta_1} = \frac{T_2}{\sin\theta_2} = \frac{T_3}{\sin\theta_3} = T,$$

for some new variable $T$. Also, since $\vec{\boldsymbol{T}}_i = T\sin\theta_i\,\hat{\boldsymbol{e}}_i$,

$$\sum_{i=1}^{3} T\sin\theta_i\,\hat{\boldsymbol{e}}_i = 0 \quad\Longrightarrow\quad \sum_{i=1}^{3}\sin\theta_i\,\hat{\boldsymbol{e}}_i = 0.$$

*Step 3: the constraint on accelerations.* Let the acceleration of O be $\vec{\boldsymbol{a}}_O$. (Remember: even though there is no net force on the massless joint, it can still accelerate.) By the rigid body condition, the relative acceleration of the two ends of a rod, along the rod, must be the centripetal acceleration that goes with their relative tangential velocity. Here the relative velocity is just $\vec{\boldsymbol{v}}_i$, since O is at rest. So, along the rod, $(\vec{\boldsymbol{a}}_i - \vec{\boldsymbol{a}}_O)\cdot\hat{\boldsymbol{e}}_i = -\frac{v_i^2}{l_i}$ (pointing towards O).

The force on $m_i$ from its rod is equal and opposite to the force of that rod on O, so the acceleration of $m_i$ is

$$\vec{\boldsymbol{a}}_i = -\frac{T_i}{m_i}\hat{\boldsymbol{e}}_i = -\frac{T\sin\theta_i}{m_i}\hat{\boldsymbol{e}}_i.$$

Putting this in, the condition becomes

$$\frac{T\sin\theta_i}{m_i} + \vec{\boldsymbol{a}}_O\cdot\hat{\boldsymbol{e}}_i = \frac{v_i^2}{l_i}$$

for all $1 \leq i \leq 3$.

*Step 4: solve.* Multiply the $i$th equation by $\sin\theta_i$ and add all three:

$$\sum_{i=1}^{3}\frac{T\sin^2\theta_i}{m_i} = \sum_{i=1}^{3}\frac{v_i^2\sin\theta_i}{l_i},$$

because the $\vec{\boldsymbol{a}}_O$ terms add up to $\vec{\boldsymbol{a}}_O\cdot\left(\sum_{i=1}^{3}\sin\theta_i\,\hat{\boldsymbol{e}}_i\right) = \vec{\boldsymbol{a}}_O\cdot\vec{\boldsymbol{0}} = 0$. So

$$T = \frac{\sum_{i=1}^{3}\dfrac{v_i^2\sin\theta_i}{l_i}}{\sum_{i=1}^{3}\dfrac{\sin^2\theta_i}{m_i}}.$$

Finally, the acceleration of the $i$th mass is simply

$$\vec{\boldsymbol{a}}_i = -\frac{T\sin\theta_i}{m_i}\hat{\boldsymbol{e}}_i = -\frac{\sum_{j=1}^{3}\dfrac{v_j^2\sin\theta_j}{l_j}}{\sum_{j=1}^{3}\dfrac{\sin^2\theta_j}{m_j}}\cdot\frac{\sin\theta_i}{m_i}\hat{\boldsymbol{e}}_i.$$

## 4.9 SYSTEMS WITH A VARYING AMOUNT OF MOVING MASS

[CP1 4.7, 6.7]

### 4.9.1 What a System Is

[CP1 4.7]

In mechanics, a system is *not* a region of space with a boundary. That wrong idea would mean that when particles enter or leave the region, we add them to or remove them from the system. Instead, a system is a *fixed set of particles*, chosen in advance and followed from then on.

So it is misleading to write the following equation for a purely translating rigid body, hoping that the $\mathrm{d}M/\mathrm{d}t$ term describes mass physically entering or leaving the body:

$$\sum \vec{F} = \frac{\mathrm{d}\vec{P}}{\mathrm{d}t} = \frac{\mathrm{d}(M\vec{v}_{\mathrm{CM}})}{\mathrm{d}t} = \frac{\mathrm{d}M}{\mathrm{d}t}\vec{v}_{\mathrm{CM}} + M\vec{a}_{\mathrm{CM}}.$$

This equation implies that at one time we are looking at one set of particles, and at another time at a completely different set! Instead, we should define an all-including system, in which different masses move at different velocities. Then we find the total momentum $p(t)$ of this all-including system as a function of time. (We use a small $p$ because we deal with a single system.) Its time derivative equals the net external force on the system.

## 4.9.2 Using the Momentum of the Whole System

[CP1 4.7]

Consider the following examples.

**Example.** [CP1 4.7, p.174] An empty box is moving with horizontal velocity $v$ on frictionless ground. You add sand, initially at rest, into the box at a rate $\sigma$ (mass added per unit time). What horizontal force is needed to keep the box moving at constant velocity?

***Solution*** Take the system to be the cart (with the sand inside it) *together with* all the sand that has been, or will be, added. After time $t$, the total horizontal momentum of this system is

$$p = (m_0 + \sigma t)v,$$

where $m_0$ is the initial mass of the cart. (The sand not yet added is at rest, so it has no momentum.) The only horizontal external force on this whole system is the force you exert on the cart. So

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \sigma v.$$

By coincidence, the equation

$$F = ma + \frac{\mathrm{d}m}{\mathrm{d}t}v$$

gives the same result, with $a = 0$ and $\mathrm{d}m/\mathrm{d}t = \sigma$. But the physical meaning of this equation is wrong, as the next example, which differs only slightly, shows.

**Example.** [CP1 4.7, p.175] An empty box is moving with velocity $v$. You hold sand of total initial mass $M$ in your hand and move it at velocity $u$ in the same direction as the box. You then release the sand into the box at a rate $\sigma$; just after leaving your hand, the sand still has velocity $u$. What force is needed to keep the box moving at constant velocity?

***Solution*** Using $F = ma + \mathrm{d}m/\mathrm{d}t\ v$ would give the same answer $\sigma v$ as before, which is clearly wrong. Instead, take the cart and all the sand together as one system. After time $t$, the masses of the cart and of the sand left in your hand are $m_0 + \sigma t$ and $M - \sigma t$. So the total horizontal momentum of the combined system is

$$p = (m_0 + \sigma t)v + (M - \sigma t)u.$$

The only net horizontal external force on this combined system is the force from you on the cart, so

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \sigma(v - u).$$

Finally, an extreme case where the $\mathrm{d}m/\mathrm{d}t$ term gives a completely unreasonable answer.

**Example.** [CP1 4.7, p.175] A box of sand moving with velocity $v$ leaks sand at a rate $\sigma$ (mass lost per unit time). There is no friction between the sand and the cart, so the sand that leaves still moves at velocity $v$. What force is needed to keep the box moving at constant velocity?

***Solution*** Using $F = ma + \mathrm{d}m/\mathrm{d}t\ v$ would give $-\sigma v$. This says a backward force is needed to stop the cart from accelerating! If we instead take the box and all the sand as one system, its total horizontal momentum does not change with time (assuming no other horizontal forces act on the sand that has left). So in fact *no* force is needed!

***Another view: two steps, and locality.*** Another way to see these problems is to split each moment into two steps. First, sand is added or lost, which may change the cart's velocity. Second, the force needed to keep the cart at constant velocity is applied.

- In the first problem, step one lowers the cart's velocity (by conservation of momentum), so a force is needed in step two to keep the velocity constant.
- In the second problem, step one lowers the cart's velocity by less, because the sand was already moving. So a smaller force is needed in step two.
- In the third problem, step one does not change the cart's velocity at all, so no force is needed.

This approach looks at the effect of adding a tiny amount of mass, and of applying a force for a tiny time. We develop it in Section 4.9.4, using the impulse-momentum theorem of Section 4.4.

This second method has one big advantage over the whole-system method: it describes the situation more precisely. When we use an all-including system, we lose detail. For example, in the second problem, once some sand has dropped, the cart's motion should not depend on how the sand still in your hand moves (you could accelerate your hand, for example). In the third problem, whether or not the sand that has left feels any force should not affect the cart's motion. In the end, interactions should only be *local*. This principle of locality is a basic pillar of physics, and casting a wider net only blurs it. But we can turn the argument round: since interactions are local, we may change in any way the parts of a system that have no direct effect on the part we care about, and the motion of that part stays the same. This justifies the assumptions we made in the problems above.

For now, let us practise finding $p$ and $\mathrm{d}p/\mathrm{d}t$ for more systems.

**EXAMPLE.** [CP1 4.7, p.176] A uniform, straight chain with linear mass density $\lambda$ and length $l$ hangs vertically at rest, with its lower tip touching the surface of a weighing scale. It is then released. The parts of the chain that hit the scale stop at once. Find the reading on the scale as a function of $x$, the vertical distance the top end of the chain has fallen. The bend at the surface of the scale is small, meaning that the length of the bend is much smaller than the spacing between the tiny masses in the chain, so the bend is effectively massless.

**Figure 4.25**

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**FIGURE 4.25.** Falling chain.

***Solution*** Take downwards as positive. The first key point is that the tension in the chain must be zero. The two tension forces on the massless bend point vertically and horizontally, so they cannot balance each other unless both are zero. So the part of the chain that has not yet hit the scale is in free fall, and its speed follows from the kinematics equation (Note 3):

$$v^2 = 2gx \quad \Longrightarrow \quad v = \sqrt{2gx}.$$

The length of chain still moving is $l - x$. The total momentum of the whole chain, including the parts already at rest on the scale, is

$$p = \lambda(l - x)\dot{x}.$$

Differentiating with the product rule,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\lambda\dot{x}^2 + \lambda(l - x)\ddot{x}.$$

Substituting $\dot{x} = v = \sqrt{2gx}$ and $\ddot{x} = g$,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -2\lambda gx + \lambda g(l - x) = -3\lambda gx + \lambda gl.$$

The net external force on the chain is its weight $\lambda gl$ (down) minus the normal force $N$ from the scale (up):

$$\begin{aligned}\lambda gl - N &= -3\lambda gx + \lambda gl\\ N &= 3\lambda gx.\end{aligned}$$

At $x = l$, the normal force suddenly drops from $3\lambda gl$ to $\lambda gl$ (the total weight of the chain). The drop happens because there are no more pieces of chain crashing into the scale. The exact size of this drop, $2\lambda gl$, is best understood with the method of Section 4.9.4, where we solve this problem again.

**EXAMPLE.** [CP1 4.7, p.178] A long, uniform chain of linear mass density $\lambda$ is coiled into a heap, initially at rest, on a horizontal, frictionless table. A tiny piece of one end hangs down through a hole in the table. There is no internal friction between parts of the chain, and the only moving part of the chain is the part below the table. Find the velocity $\dot{x}$ of the moving part as a function of the length $x$ that has fallen below the table, and then find $x(t)$.

**Figure 4.26**

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**FIGURE 4.26.** Falling chain from a heap.

***Solution*** Only the hanging part moves, so the momentum of the whole chain is

$$p = \lambda x\dot{x}.$$

The net vertical external force on the whole chain is $\lambda gx$, the weight of the hanging part. (For the part on the table, the normal force and the weight cancel exactly, since its vertical momentum stays zero.) Also, there is no tension in the chain: the chain on the frictionless table cannot feel a net horizontal tension from the piece in the hole. So

$$\lambda gx = \frac{\mathrm{d}p}{\mathrm{d}t} = \lambda\dot{x}^2 + \lambda x\ddot{x}.$$

Use the trick $\ddot{x} = \frac{1}{2}\,\mathrm{d}\dot{x}^2/\mathrm{d}x$ (it follows from $\ddot{x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}$). Dividing by $\frac{1}{2}\lambda x$,

$$\frac{\mathrm{d}\dot{x}^2}{\mathrm{d}x} + \frac{2}{x}\dot{x}^2 = 2g.$$

Multiply by the integrating factor $x^2$ (Note 2). The left side becomes an exact derivative:

$$\begin{aligned}x^2\frac{\mathrm{d}\dot{x}^2}{\mathrm{d}x} + 2x\dot{x}^2 &= 2gx^2\\ \frac{\mathrm{d}(x^2\dot{x}^2)}{\mathrm{d}x} &= 2gx^2\\ \int_0^{x^2\dot{x}^2}\mathrm{d}(x^2\dot{x}^2) &= \int_0^x 2gx^2\,\mathrm{d}x\\ x^2\dot{x}^2 &= \frac{2}{3}gx^3\\ \dot{x} &= \sqrt{\frac{2}{3}gx},\end{aligned}$$

where we reject the negative root because the falling part can only move downwards (positive). This follows from the fact that the net force on the system is downwards. Then, separating variables,

$$\int_0^x \frac{1}{\sqrt{x}}\,\mathrm{d}x = \int_0^t \sqrt{\frac{2}{3}g}\,\mathrm{d}t$$
$$2\sqrt{x} = \sqrt{\frac{2}{3}g}\,t$$
$$x = \frac{1}{6}gt^2.$$

**Example.** [CP1 4.7, p.179] A uniform, straight chain of total mass $m$ and length $l$ lies at rest on a horizontal, frictionless table. At the start, a tiny piece at one end of the chain passes through a hole in the table and hangs vertically. Parts of the chain never "overshoot" the hole. Find the velocity $\dot{x}$ of the chain as a function of the length $x$ that has fallen below the table, and then find $x(t)$. You will notice an error in $x(t)$. What causes this error, and why did it not happen in the previous problem?

Figure 4.27

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Figure 4.27. Falling straight chain.

***Solution*** This time the whole chain moves with the same speed $\dot{x}$. Treat the two parts separately.

The tension $T$ in the piece of chain inside the hole changes the horizontal momentum of the part on the table, $\frac{m}{l}(l-x)\dot{x}$:

$$T = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{l}(l-x)\dot{x}\right).$$

On the other hand, the weight of the hanging part, $\frac{mg}{l}x$, minus this tension changes the vertical momentum of the hanging part, $\frac{m}{l}x\dot{x}$:

$$\frac{mg}{l}x - T = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{l}x\dot{x}\right).$$

Adding the two equations (the $T$ cancels, and the two momenta add up to $m\dot{x}$),

$$\frac{mg}{l}x = \frac{\mathrm{d}(m\dot{x})}{\mathrm{d}t} = m\ddot{x}.$$

Using $\ddot{x} = \dot{x}\,\mathrm{d}\dot{x}/\mathrm{d}x$ and separating variables,

$$\int_0^{\dot{x}} \dot{x}\,\mathrm{d}\dot{x} = \int_0^x \frac{g}{l}x\,\mathrm{d}x$$
$$\frac{\dot{x}^2}{2} = \frac{gx^2}{2l}$$
$$\dot{x} = \sqrt{\frac{g}{l}}\,x.$$

We choose the positive root, since $x$ starts slightly positive and $\ddot{x}$ is proportional to $x$ throughout the motion. Now try to separate variables and integrate:

$$\int_{0^+}^x \frac{1}{x}\,\mathrm{d}x = \int_0^t \sqrt{\frac{g}{l}}\,\mathrm{d}t$$
$$\ln x = \sqrt{\frac{g}{l}}\,t + \ln 0^+.$$

An awkward $\ln 0^+$ term appears out of nowhere! It is there because the chain needs an *infinite* time to go from a negligible hanging length to a noticeable one (so $\ln x$ goes to minus infinity).

This defect appears here because the chain is not slack. At the start, the tension in the chain equals the weight of the tiny hanging piece (so that this piece does not have an infinite acceleration). This small tension pulls the part of the chain on the table towards the hole, giving it momentum. Since that part is massive, it naturally takes an infinite time to pull it into the hole by a noticeable amount (and to give it a noticeable momentum).

In the previous problem, the whole chain was slack, so no force was needed to drag the heap on the table into the hole. The heap simply rearranges itself while its center of mass stays in place. Also, at the start, the weight of the tiny hanging piece goes entirely into increasing that piece's own momentum, with no tension holding it back. Once it gets past this first hurdle and reaches a noticeable hanging length, the hanging part keeps a self-sustaining cycle going: a larger $x$ leads to a larger $\dot{x}$. (This also happens for the straight chain, but the straight chain fails at the first hurdle.) In terms of momentum: the weight of the (at first tiny) hanging part only has to increase its *own* momentum, not that of the whole chain, so it can reach a large velocity.

Despite all this, the result for $\dot{x}(x)$ in this problem is still correct. It just takes an infinite time to reach a noticeable $x$.

## 4.9.3 Throwing Mass Away with No External Force

[CP1 6.7]

***A human-powered boat.*** You and your $(N-1)$ friends have a brilliant idea: a boat pushed along by people! Each of you has the same mass $m$. At first, all $N$ of you stand on a boat at rest, whose mass we take to be negligible. Each of you is only strong enough to run along the boat and jump off at velocity $u$ *relative to the boat*. How should $(N-1)$ people jump so that the last person moves as fast as possible? The boat rests on frictionless ground.

*All jump together.* Suppose all $(N-1)$ people jump off at the same time. Let the final velocity of the boat be $v$. The jumpers then move at $v-u$ in the lab frame. By conservation of momentum (the total was zero),

$$0 = (N-1)m(v-u) + mv \quad\Longrightarrow\quad v = \frac{N-1}{N}u.$$

*Jump one at a time.* Now suppose the $(N-1)$ people jump off one at a time. Let $v_i$ be the speed of the boat after $i$ people have jumped. Look at the moment when the $(i+1)$th person jumps. Before the jump, the total momentum of the boat and the people on it is $(N-i)mv_i$. After the jump, the boat moves at $v_{i+1}$, so the person who jumped moves at $v_{i+1} - u$ in the lab frame (Fig. 4.28).

**Figure 4.28**

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FIGURE 4.28. Propelling a boat.

By conservation of momentum,

$$(N-i)mv_i = (N-i-1)mv_{i+1} + m(v_{i+1} - u).$$

The right side is $(N-i)mv_{i+1} - mu$, so

$$v_{i+1} = v_i + \frac{1}{N-i}u.$$

Using this recursion formula again and again, starting from $v_0 = 0$,

$$v_{N-1} = \left(\frac{1}{2} + \cdots + \frac{1}{N}\right)u = u\sum_{i=2}^{N}\frac{1}{i}.$$

This is larger than the previous final velocity. There are $(N-1)$ terms, and each is equal to or larger than $\frac{u}{N}$, so the sum is at least $\frac{N-1}{N}u$.

The second method is in fact the best one. The reason: people who have jumped off move at $-u$ relative to the boat's velocity *after* their jump. If they jump one by one, each person leaves while the boat is still slower (it has not yet been sped up by the later jumps). So each person leaves with a larger velocity towards the left in the lab frame than when all jump at once. Since the total momentum of the system must still be conserved, the last person and the boat end up moving faster.

*Smaller pieces.* Finally, suppose there are now $j(N-1)$ people of mass $\frac{m}{j}$ each, while the last person still has mass $m$ (so the total mass is unchanged). Repeat the calculation with people jumping one at a time. The final velocity of the last person on the boat is

$$\begin{aligned} v_{jN-1} &= \left(\frac{1}{j+1} + \cdots + \frac{1}{jN}\right)u = u\sum_{i=j+1}^{jN}\frac{1}{i} \\ &> \left(j\cdot\frac{1}{2j} + j\cdot\frac{1}{3j} + \cdots + j\cdot\frac{1}{jN}\right)u. \end{aligned}$$

The inequality comes from splitting the terms into groups of $j$ consecutive terms: each term in a group is at least as big as the last (smallest) term of that group. The last expression is just $v_{N-1}$ found before. So it also pays to split the thrown mass into smaller pieces, as well as throwing the pieces one at a time.

***Fuel-propelled rocket.*** Since throwing many small masses one after another gives the boat the most speed, let us replace the boat of people by a rocket of initial mass $m_i$. Now and then (not necessarily at regular times), it throws out a tiny amount of mass from its back, at velocity $u$ *relative to itself*. What is the speed of the rocket when its mass has fallen to $m_f$, if its initial speed is $v_i$?

**Figure 4.29**

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FIGURE 4.29. Rocket.

Let the mass and speed of the rocket at some moment be $m$ and $v$. The rocket throws out an amount $-\mathrm{d}m$ of mass. ($\mathrm{d}m$ is the change in the rocket's mass, which is negative, so $-\mathrm{d}m$ is positive.) After this, the rocket's mass is $m + \mathrm{d}m$ and its velocity in the lab frame is $v + \mathrm{d}v$. So the ejected fuel moves at $v + \mathrm{d}v - u$ (Fig. 4.29). By conservation of momentum,

$$mv = (m + \mathrm{d}m)(v + \mathrm{d}v) - \mathrm{d}m\,(v + \mathrm{d}v - u).$$

On the right, the terms $\mathrm{d}m\,(v + \mathrm{d}v)$ cancel, leaving $mv + m\,\mathrm{d}v + u\,\mathrm{d}m$. So

$$\mathrm{d}v = -\frac{u}{m}\,\mathrm{d}m\,.$$

Integrating from the initial state to the final state,

$$\begin{aligned} v_f - v_i &= -u\int_{m_i}^{m_f}\frac{1}{m}\,\mathrm{d}m = -u\Big[\ln|m|\Big]_{m_i}^{m_f} = u\ln\frac{m_i}{m_f}, \\ v_f &= v_i + u\ln\frac{m_i}{m_f}. \end{aligned}$$

This rocket example shows how to use conservation of momentum for such problems. Sadly, the logarithm in the result is very discouraging, especially since this way of propelling is the most efficient one possible!

### 4.9.4 Systems under a Net External Force

[CP1 6.7]

The boat and the rocket had no net external force on them, so we could use conservation of momentum. For systems with varying mass *under* a net external force, we can apply the impulse-momentum theorem (Eq. 4.11) to the system over a tiny time interval $\mathrm{d}t$. This is the "local" method described earlier.

EXAMPLE. [CP1 6.7, p.317] A cart is moving at velocity $v$. You start to add sand, moving at velocity $u$, to the cart at a rate $\sigma$ (mass per unit time). Find the force $F$ you must exert on the cart so that it moves at constant velocity $v$.

(This is the situation of the second sand-box example above, now solved with the local method.)

***Solution*** Let the mass of the cart at this moment be $m$. During a time $\mathrm{d}t$, the cart gains an extra mass $\mathrm{d}m$ of sand, which was moving at velocity $u$. The cart's mass and velocity become $m + \mathrm{d}m$ and $v + \mathrm{d}v$. By the impulse-momentum theorem, applied to the cart plus this bit of sand,

$$F\,\mathrm{d}t = \Delta p = (m + \mathrm{d}m)(v + \mathrm{d}v) - (mv + u\,\mathrm{d}m).$$

$F\,\mathrm{d}t$ is the impulse delivered in the short time, and the right side is the change in the combined momentum of the cart and the incoming sand. Drop the second-order term $\mathrm{d}m\,\mathrm{d}v$ and divide the whole equation by $\mathrm{d}t$:

$$F = \frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}v}{\mathrm{d}t}.$$

Since $\mathrm{d}v/\mathrm{d}t = 0$ and $\mathrm{d}m/\mathrm{d}t = \sigma$,

$$F = \sigma(v - u).$$

**Example.** [CP1 6.7, p.318] A chain of uniform linear mass density $\lambda$ and length $l$ is held at rest, hanging vertically, with its lower end just touching a table. It is then released. Find the normal force of the table on the chain as a function of $x$, the distance the top of the chain has fallen. When a part of the chain hits the table, it stops at once.

(This is the falling chain solved above by the whole-system method; here we use the local method.)

Figure 4.30

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**Figure 4.30.** Falling chain.

***Solution*** The normal force of the table on the chain has two jobs. First, it holds up the weight of the part of the chain already at rest on the table. Second, it must stop the part of the falling chain that hits the table. We find the force $F$ needed for the second job, and then add the weight of the chain already on the table.

In a time $\mathrm{d}t$, a mass $\lambda v\,\mathrm{d}t$ of chain, moving at $v$, comes to rest. (The extra velocity that this falling piece gains from gravity during $\mathrm{d}t$ is second order, so we ignore it.) By the impulse-momentum theorem,

$$F\,\mathrm{d}t = v\,\mathrm{d}m = \lambda v^2\,\mathrm{d}t\,.$$

We can write $v$ in terms of $x$ using conservation of energy (Note 6) or the kinematics equations (Note 3), since the moving part of the chain is in free fall: there is no tension in the chain (see the whole-system solution above).

$$v^2 = 2gx \quad\Longrightarrow\quad F = 2\lambda gx.$$

Finally, add back the weight $\lambda gx$ of the part of the chain at rest on the table to get the normal force:

$$N = F + \lambda gx = 3\lambda gx.$$

This approach makes clear why $N$ suddenly drops from $3\lambda gl$ to $\lambda gl$ at $x = l$: there is no longer any falling piece hitting the scale, and at $x = l$ that part contributed $2\lambda gl$ to $N$.

Sometimes the change in mass can be written in terms of a distance travelled. Then the *work-energy theorem* (Note 6) can be applied over a tiny distance, as the next example shows.

**Example.** [CP1 6.7, p.319] A massless bucket holds a mass $M$ of sand and is at rest at the origin. You pull it in the positive $x$-direction with a constant tension $T$ across frictionless ground. The bucket leaks sand at the rate

$$\frac{\mathrm{d}m}{\mathrm{d}x} = -\frac{M}{L},$$

where $m$ is the mass of the bucket (with its sand) at that moment. Find its kinetic energy as a function of its coordinate $x$, for $x < L$. ("An Introduction to Classical Mechanics")

***Solution*** Let $E(x)$ be the kinetic energy of the bucket and its sand at that moment. Look at the change $\mathrm{d}E$ in kinetic energy as the bucket moves a tiny distance $\mathrm{d}x$. Two things change it:

- By the work-energy theorem, the bucket gains kinetic energy $T\,\mathrm{d}x$ from the work done on it by the tension.
- It also "gains" kinetic energy $\frac{\mathrm{d}m}{m}E$ as its mass changes by $\mathrm{d}m$. This quantity is negative, because $\mathrm{d}m$ is negative: the leaked sand carries away its share, the fraction $\frac{|\mathrm{d}m|}{m}$, of the kinetic energy.

So

$$\mathrm{d}E = T\,\mathrm{d}x + \frac{\mathrm{d}m}{m}E.$$

Dividing by $\mathrm{d}x$,

$$\frac{\mathrm{d}E}{\mathrm{d}x} = T + \frac{\mathrm{d}m}{\mathrm{d}x}\cdot\frac{E}{m}.$$

Since $m = M - \frac{M}{L}x$,

$$\frac{\mathrm{d}m}{\mathrm{d}x}\cdot\frac{E}{m} = -\frac{M}{L}\cdot\frac{E}{M - \frac{M}{L}x} = -\frac{E}{L - x},$$

and so

$$\frac{\mathrm{d}E}{\mathrm{d}x} + \frac{E}{L - x} = T.$$

Multiply by the integrating factor $\frac{1}{L-x}$. The left side becomes an exact derivative:

$$\frac{1}{L - x}\frac{\mathrm{d}E}{\mathrm{d}x} + \frac{E}{(L - x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{E}{L - x}\right) = \frac{T}{L - x}$$

$$\int_0^{\frac{E}{L-x}} \mathrm{d}\left(\frac{E}{L - x}\right) = \int_0^x \frac{T}{L - x}\,\mathrm{d}x\,.$$

The lower limit is 0 because the bucket starts at rest ($E = 0$ at $x = 0$). Since $\int_0^x \frac{\mathrm{d}x}{L-x} = \ln\frac{L}{L-x}$, simplifying gives

$$E = T(L - x)\ln\frac{L}{L - x}.$$

# PROBLEMS

## Center of Mass

**1. Regular Pentagon*** [CP1 P4.1]

Find the center of mass of a uniform, regular pentagon with edge length $l$.

***Solution*** Draw two neighboring lines of symmetry of the pentagon. By symmetry (Section 4.5), the center of mass lies on both, so it is where they cross. The interior angle of a pentagon is

$$\frac{3\pi}{5}.$$

A line of symmetry through a vertex cuts that angle in half, so the lines from the center to the two ends of an edge make angle $\frac{3\pi}{10}$ with the edge. By simple trigonometry, the perpendicular distance from the center of mass to the edge that a line of symmetry cuts is

$$\frac{l}{2}\tan\frac{3\pi}{10}.$$

**2. A Strange Rod*** [CP1 P4.2]

Find the center of mass of a one-dimensional rod of length $l$ whose linear mass density at a point is $\lambda = \frac{k}{x+l}$, where $k$ is a constant and $x$ is the distance from the left end of the rod to that point.

***Solution*** Put the origin at the left end of the rod. Write $\frac{x}{x+l} = 1 - \frac{l}{x+l}$:

$$\begin{aligned}\int x\,\mathrm{d}m &= \int_0^l x\cdot\frac{k}{x+l}\,\mathrm{d}x = \int_0^l k\left(1-\frac{l}{x+l}\right)\mathrm{d}x\\ &= \big[kx - kl\ln|x+l|\big]_0^l = kl - kl\ln 2,\\ \int \mathrm{d}m &= \int_0^l \frac{k}{x+l}\,\mathrm{d}x = \big[k\ln|x+l|\big]_0^l = k\ln 2,\\ x_{\text{CM}} &= \frac{\int x\,\mathrm{d}m}{\int \mathrm{d}m} = l\left(\frac{1}{\ln 2}-1\right).\end{aligned}$$

**3. Cone**** [CP1 P4.3]

A uniform cone has base area $A$ and height $h$. Find how far below its vertex its center of mass lies.

***Solution*** Let the density of the cone be $\rho$. Put the origin at the vertex, with the positive $z$-axis along the axis of the cone, meeting the base at right angles. The cross-section between $z$ and $z + \mathrm{d}z$ is a disk. Its linear size grows in proportion to $z$, so its area is $\frac{z^2}{h^2}A$ and its volume is $\frac{z^2}{h^2}A\,\mathrm{d}z$. Then

$$\begin{aligned}z_{\text{CM}} &= \frac{1}{M}\int_0^h z\,\mathrm{d}m = \frac{1}{M}\int_0^h \frac{\rho z^3}{h^2}A\,\mathrm{d}z\\ &= \frac{1}{M}\left[\frac{\rho z^4}{4h^2}A\right]_0^h = \frac{Ah^2\rho}{4M}.\end{aligned}$$

Also,

$$M = \frac{1}{3}Ah\rho \quad\Longrightarrow\quad z_{\text{CM}} = \frac{3}{4}h.$$

*Another method.* Take a thin isosceles triangle of tiny width and thickness, whose axis of symmetry makes the half angle of the cone with the $z$-axis. Turning it through a complete revolution about the $z$-axis gives a thin conical layer (a hollow cone). So the center of mass of this conical layer is at a height $\frac{l}{3}$ above its base, where $l$ is the layer's height (the triangle's center of mass is one-third of its height from its base). Now we can build the solid cone of height $h$ by stacking conical layers with heights from $l = 0$ to $l = h$ inside each other. So the center of mass of the solid cone is the weighted average of the centers of mass of these layers. The weight of each layer is its mass, which is proportional to its surface area and hence to its height squared, $l^2$. So the center of mass of a solid cone of height $h$ lies at a height

$$\frac{\int_0^h l^2\cdot\frac{l}{3}\,\mathrm{d}l}{\int_0^h l^2\,\mathrm{d}l} = \frac{h}{4}$$

above its base, on its axis (by symmetry).

**4. Spherical Cap**** [CP1 P4.4]

A spherical cap of height $h$ is cut from a uniform sphere of radius $R$ (a spherical cap is what you get by slicing a sphere with a plane). Show that its center of mass is

$$\frac{3(2R-h)^2}{4(3R-h)}$$

above the center of the original sphere.

***Solution*** Put the origin at the center of the original sphere, with the positive $z$-axis through the top (vertex) of the cap. By symmetry, the center of mass lies on the $z$-axis. Take thin disks between $z$ and $z + \mathrm{d}z$. The radius $r$ of such a disk obeys

$$r^2 + z^2 = R^2 \quad\Longrightarrow\quad r^2 = R^2 - z^2.$$

So the volume of the disk is $\pi r^2\,\mathrm{d}z = \pi(R^2 - z^2)\,\mathrm{d}z$, and

$$\begin{aligned}z_{\text{CM}} &= \frac{1}{M}\int_{R-h}^R z\,\mathrm{d}m = \frac{1}{M}\int_{R-h}^R \rho\pi(R^2z - z^3)\,\mathrm{d}z\\ &= \frac{\rho\pi h^2(2R-h)^2}{4M}.\end{aligned}$$

(The integral is $\left[\frac{R^2z^2}{2} - \frac{z^4}{4}\right]_{R-h}^R = \frac{1}{4}\left[R^2 - (R-h)^2\right]^2 = \frac{1}{4}h^2(2R-h)^2$.) The volume of the whole cap is

$$V = \int_{R-h}^R \pi(R^2 - z^2)\,\mathrm{d}z = \frac{\pi h^2(3R-h)}{3}, \qquad M = \rho V.$$

Therefore

$$z_{\text{CM}} = \frac{3(2R-h)^2}{4(3R-h)}.$$

**5. Triangle**** [CP1 P4.5]

Write the coordinates of the center of mass of a general uniform triangle in terms of the coordinates of its three vertices. Hint: use the result for a right-angled triangle.

A *plated triangle* is made as follows. Take a uniform triangle of surface mass density $\sigma$, with vertices at $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$, and join the midpoints of its edges. Cover the new triangle with a plating of surface mass density $\sigma$. Repeat this with the new triangle and with all later triangles. Find the center of mass of this plated triangle.

As a separate question, prove geometrically that the center of mass of a uniform triangle is the point where its three medians meet. (You do not need to prove that the medians meet at one point.)

***Solution*** Let the plane of the triangle be the $x'y'$-plane. Put the origin O′ at one vertex, and turn the $x'$-axis so that a second vertex lies on it, at $(x'_2, 0)$. The last vertex is at $(x'_3, y'_3)$.

**Figure 4.31**

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FIGURE 4.31. Triangle.

The foot of the perpendicular from the third vertex to the $x'$-axis is at $(x'_3, 0)$. (In Fig. 4.31 it lies outside the triangle, but its exact position does not matter.) Now use the two right-angled triangles formed: one with vertices O′, $(x'_3, 0)$, $(x'_3, y'_3)$, and another with vertices $(x'_2, 0)$, $(x'_3, 0)$, $(x'_3, y'_3)$. Here we must first "fill the hole" to get the big triangle, and then take away the small triangle. By the result for a right-angled triangle, their centers of mass are at

$$\left(\tfrac{2}{3}x'_3,\ \tfrac{1}{3}y'_3\right) \quad \text{and} \quad \left(\tfrac{1}{3}x'_2 + \tfrac{2}{3}x'_3,\ \tfrac{1}{3}y'_3\right).$$

Let the mass density take the value 2, so that each triangle's mass equals base times height. Their masses are then $x'_3 y'_3$ and $-(x'_3 - x'_2)y'_3$ (remember the minus sign for the second one, since we are subtracting it). So the center of mass of the general triangle is

$$\begin{pmatrix} x'_{\text{CM}} \\ y'_{\text{CM}} \end{pmatrix} = \frac{x'_3 y'_3 \begin{pmatrix} \frac{2}{3}x'_3 \\ \frac{1}{3}y'_3 \end{pmatrix} - (x'_3 - x'_2)y'_3 \begin{pmatrix} \frac{1}{3}x'_2 + \frac{2}{3}x'_3 \\ \frac{1}{3}y'_3 \end{pmatrix}}{x'_3 y'_3 - (x'_3 - x'_2)y'_3}$$
$$= \begin{pmatrix} \frac{1}{3}x'_2 + \frac{1}{3}x'_3 \\ \frac{1}{3}y'_3 \end{pmatrix}.$$

In terms of the position vectors $\vec{r}\,'_2$ and $\vec{r}\,'_3$ of the second and third vertices relative to O′,

$$\vec{r}\,'_{\text{CM}} = \frac{1}{3}\vec{r}\,'_2 + \frac{1}{3}\vec{r}\,'_3.$$

Now take a general origin O, where the vertices have position vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$. Note that $\vec{r}_{\text{CM}} = \vec{r}\,'_{\text{CM}} + \vec{r}_1$, with $\vec{r}\,'_2 = \vec{r}_2 - \vec{r}_1$ and $\vec{r}\,'_3 = \vec{r}_3 - \vec{r}_1$. So

$$\vec{r}_{\text{CM}} = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3).$$

For the plated triangle: the midpoints of the edges of a triangle with vertices at $\vec{r}_1, \vec{r}_2, \vec{r}_3$ have position vectors $\frac{\vec{r}_1+\vec{r}_2}{2}$, $\frac{\vec{r}_1+\vec{r}_3}{2}$ and $\frac{\vec{r}_2+\vec{r}_3}{2}$. So the center of mass of the new triangle formed by joining the midpoints is

$$\vec{r}_{\text{CM,new}} = \frac{1}{3}\left(\frac{\vec{r}_1 + \vec{r}_2}{2} + \frac{\vec{r}_1 + \vec{r}_3}{2} + \frac{\vec{r}_2 + \vec{r}_3}{2}\right)$$
$$= \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) = \vec{r}_{\text{CM,old}},$$

the same as the center of mass of the original triangle. Every added layer has this same center of mass, so the center of mass of the plated triangle is just that of the first triangle:

$$\left(\frac{x_1 + x_2 + x_3}{3},\ \frac{y_1 + y_2 + y_3}{3}\right).$$

**Figure 4.32**

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FIGURE 4.32. Triangle and median.

For the last part, take one median, BM, in Fig. 4.32. Take two lines, $\mathrm{D_1D_2}$ and $\mathrm{E_1E_2}$, parallel to BM and at equal distances from it on opposite sides. Since $\triangle AD_1D_2 \sim \triangle ABM$, $\triangle CE_1E_2 \sim \triangle CBM$, and $\overline{AM} = \overline{MC}$ (by the definition of a median), we get $\overline{D_1D_2} = \overline{E_1E_2}$. So the thin strips along $\mathrm{D_1D_2}$ and $\mathrm{E_1E_2}$ have equal masses. They are also at equal perpendicular distances ($y$) from BM, but on opposite sides. So if we take an axis along BM and measure the coordinate (call it $z$) perpendicular to BM, the two strips together add nothing to the $z$-coordinate of the center of mass. Repeating this for all pairs of parallel strips at equal distances from BM, the $z$-coordinate of the center of mass of the whole triangle is zero: the center of mass lies on BM. Doing the same for the other two medians proves that the center of mass is where the medians meet.

**6. Cylindrical Segment***** [CP1 P4.6]

Find the position of the center of mass of the cylindrical segment below. It has uniform density and a circular base of radius $r$, and is obtained by slicing a cylinder with a plane.

**Figure 4.33**

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Figure 4.33. Cylindrical segment (problem figure).

***Solution*** We use Cartesian coordinates, with the origin at the left tip of the segment (side view, Fig. 4.34). In Cartesian coordinates, $\mathrm{d}m$ is just $\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where $\rho$ is the density of the wedge. Take a tiny element at $(x, y, z)$.

**Figure 4.34**

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Figure 4.34. Cylindrical segment.

We must be very careful with the limits of integration for $x$, $y$ and $z$. For a given $x$, $z$ runs from 0 to $x\tan\theta$, while $y$ runs from $-\sqrt{r^2-(r-x)^2}$ to $\sqrt{r^2-(r-x)^2}$ (the half-width of the circular base at that $x$). By symmetry, the $y$-coordinate of the center of mass is 0. We compute the other two coordinates.

$$\begin{aligned}\int x\,\mathrm{d}m &= \int_0^{2r}\int_{-\sqrt{r^2-(r-x)^2}}^{\sqrt{r^2-(r-x)^2}}\int_0^{x\tan\theta} x\rho\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\ &= \rho\int_0^{2r}\int_{-\sqrt{r^2-(r-x)^2}}^{\sqrt{r^2-(r-x)^2}} x^2\tan\theta\,\mathrm{d}y\,\mathrm{d}x\\ &= \rho\tan\theta\int_0^{2r} 2x^2\sqrt{r^2-(r-x)^2}\,\mathrm{d}x\,.\end{aligned}$$

The last integral can be done with the substitution $r-x = r\sin\varphi$, so $\mathrm{d}x = -r\cos\varphi\,\mathrm{d}\varphi$, $x^2 = r^2 - 2r^2\sin\varphi + r^2\sin^2\varphi$, and $\sqrt{r^2-(r-x)^2} = r\cos\varphi$. As $x$ goes from 0 to $2r$, $\varphi$ goes from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\begin{aligned}&\int_0^{2r} x^2\sqrt{r^2-(r-x)^2}\,\mathrm{d}x\\ &= -\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}(r^2-2r^2\sin\varphi+r^2\sin^2\varphi)\,r\cos\varphi\cdot r\cos\varphi\,\mathrm{d}\varphi\\ &= \frac{5}{8}\pi r^4.\end{aligned}$$

(Swap the limits to remove the minus sign. Over $-\frac{\pi}{2}$ to $\frac{\pi}{2}$: $\int\cos^2\varphi\,\mathrm{d}\varphi = \frac{\pi}{2}$, the $\sin\varphi\cos^2\varphi$ term gives 0, and $\int\sin^2\varphi\cos^2\varphi\,\mathrm{d}\varphi = \frac{\pi}{8}$.) Thus

$$\int x\,\mathrm{d}m = \frac{5}{4}\rho\tan\theta\,\pi r^4.$$

Similarly,

$$\begin{aligned}\int z\,\mathrm{d}m &= \int_0^{2r}\int_{-\sqrt{r^2-(r-x)^2}}^{\sqrt{r^2-(r-x)^2}}\int_0^{x\tan\theta} z\rho\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\ &= \rho\int_0^{2r}\int_{-\sqrt{r^2-(r-x)^2}}^{\sqrt{r^2-(r-x)^2}}\frac{1}{2}x^2\tan^2\theta\,\mathrm{d}y\,\mathrm{d}x\\ &= \rho\tan^2\theta\int_0^{2r} x^2\sqrt{r^2-(r-x)^2}\,\mathrm{d}x\\ &= \frac{5}{8}\rho\tan^2\theta\,\pi r^4.\end{aligned}$$

Finally, the volume of the segment is easy to find: two such segments fit together to make a cylinder of radius $r$ and height $2r\tan\theta$. So the total mass of the segment is

$$M = \rho\pi r^3\tan\theta,$$

and

$$\begin{aligned}x_{\mathrm{CM}} &= \frac{\int x\,\mathrm{d}m}{M} = \frac{5}{4}r, \qquad y_{\mathrm{CM}} = 0,\\ z_{\mathrm{CM}} &= \frac{\int z\,\mathrm{d}m}{M} = \frac{5}{8}r\tan\theta.\end{aligned}$$

**7. Constant Ratio***** [CP1 P4.7]

A rod of length $l$ and mass $M$ lies along the $x$-axis, with ends at $x = 0$ and $x = l$. The rod has a special property: if we cut it at any point $x = y$ and keep the part between $x = 0$ and $x = y$, the center of mass of this remaining part is at $ky$, where $k$ is a constant. Find $\lambda(x)$, the linear mass density of the rod as a function of $x$.

***Solution*** In mathematical form, the property is

$$\frac{\int_0^y \lambda x\,\mathrm{d}x}{y\int_0^y \lambda\,\mathrm{d}x} = k.$$

Let $\int_0^y \lambda x\,\mathrm{d}x = g(y)$ and $\int_0^y \lambda\,\mathrm{d}x = m(y)$. Then

$$g(y) = kym(y).$$

Differentiate with respect to $y$. By the fundamental theorem of calculus, $g'(y) = \lambda(y)y$ and $m'(y) = \lambda(y)$, so

$$\lambda(y)y = km(y) + ky\lambda(y).$$

Rearranging,

$$\frac{\lambda(y)}{m(y)} = \frac{k}{(1-k)y}.$$

Since $\lambda(y) = \mathrm{d}m/\mathrm{d}y$, the left side is $\mathrm{d}(\ln m)/\mathrm{d}y$, so

$$\int \mathrm{d}(\ln m) = \int\frac{k}{(1-k)y}\,\mathrm{d}y\,.$$

Integrating and simplifying,

$$m(y) = Ay^{\frac{k}{1-k}},$$

where $A$ is a constant. We find $A$ from $m(l) = M$:

$$A = \frac{M}{l^{\frac{k}{1-k}}}.$$

So

$$m(y) = \frac{M}{l^{\frac{k}{1-k}}} y^{\frac{k}{1-k}},$$
$$\lambda(x) = \frac{\mathrm{d}m(x)}{\mathrm{d}x} = \frac{kM}{(1-k)\,l^{\frac{k}{1-k}}} x^{\frac{2k-1}{1-k}}.$$

A handy limiting case to check is $k = \frac{1}{2}$, which should be a rod of constant density. It is, since the exponent becomes

$$\frac{2k-1}{1-k} = \frac{2 \cdot \frac{1}{2} - 1}{1 - \frac{1}{2}} = 0.$$

**8. Square Fractal***** [CP1 P4.8]

Find the center of mass of the square fractal below, whose surface mass density is uniform. The largest square has side $l$, and each later square has half the side of the one before it. Note that the fractal only "grows" in one direction.

**Figure 4.35**

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**Figure 4.35.** Square fractal (problem figure).

***Solution*** Let the center of mass be a horizontal distance $2x$ from the left edge of the largest square. (Its vertical position is obvious by symmetry.)

The fractal is made of a smaller copy of the fractal, two squares of side $\frac{l}{2}$, and the largest square of side $l$. By scaling, the center of mass of the smaller fractal is a distance $x$ from the left edge of *its* largest square. We only need the mass of the smaller fractal. Taking the mass density as one (its value does not matter),

$$\frac{l^2}{4} + \frac{3l^2}{16} + \frac{3l^2}{64} + \cdots = \frac{l^2}{2}.$$

(After the first term, this is a geometric series: $\frac{3l^2}{16} \cdot \frac{1}{1-\frac{1}{4}} = \frac{l^2}{4}$.) The largest square and the two squares of side $\frac{l}{2}$ have total mass $l^2 + 2 \cdot \frac{l^2}{4} = \frac{3l^2}{2}$, with their center of mass at $\frac{l}{2}$; the smaller fractal's center of mass is at $l + x$. Now compute the center of mass of the original fractal from these parts:

$$\frac{\frac{l}{2} \cdot \frac{3l^2}{2} + (l + x) \cdot \frac{l^2}{2}}{\frac{3l^2}{2} + \frac{l^2}{2}} = 2x.$$

Solving,

$$2x = \frac{5}{7} l.$$

There is an even neater way to get the mass ratios. By scaling, the original fractal has four times the mass of the smaller fractal, since corresponding squares have areas in the ratio $4:1$. So the three remaining squares have three times the mass of the smaller fractal. Then

$$\frac{\frac{l}{2} \cdot 3 + (l + x) \cdot 1}{3 + 1} = 2x,$$

and again $2x = \frac{5}{7} l$.

**9. Triangle Fractal***** [CP1 P4.9]

Take an equilateral triangle of side $l$. Fill with mass the equilateral triangle inside it whose vertices are the midpoints of its sides (it covers a quarter of the area). This leaves three empty equilateral triangles of side $\frac{l}{2}$. The bottom two of these then go through the same process, and so on for all later triangles, forever. Find the vertical distance between the center of mass of this fractal and the bottom edge of the original triangle, in the figure below.

**Figure 4.36**

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**Figure 4.36.** Triangle fractal (problem figure).

***Solution*** Let the vertical distance between the center of mass of the fractal and the bottom edge be $2y$. The original fractal is made of two smaller copies of the fractal, with all lengths halved, and one equilateral triangle of side $\frac{l}{2}$. By scaling, the center of mass of each smaller fractal is a distance $y$ above its bottom edge, which lies on the bottom edge of the original triangle.

We only need the mass ratios. By scaling, the original fractal has four times the mass of a smaller fractal. So the equilateral triangle of side $\frac{l}{2}$ has twice the mass of a smaller fractal. This middle triangle points downwards, with its apex on the bottom edge and its top edge at height $\frac{\sqrt{3}}{4} l$, so its center of mass is at height $\frac{\sqrt{3}}{6} l$. Then

$$\frac{y \cdot 1 + y \cdot 1 + \frac{\sqrt{3}}{6} l \cdot 2}{1 + 1 + 2} = 2y \quad \Longrightarrow \quad 2y = \frac{\sqrt{3}}{9} l.$$

## Systems with No Constraints

**10. Walking on a Plank*** [CP1 P4.10]

A wooden plank of length $l$ and mass $M$ lies on frictionless horizontal ground. Consider the one-dimensional problem in which a person of mass $m$ starts at one end of the plank and walks to the other end. Find the horizontal displacement of the plank if the plank and the person were both at rest at the start.

Now suppose a massless ant rests at one end of the plank. The ant has a supply of snowballs of total mass $m$. The ant starts throwing snowballs at velocity $v$, and they stick to a massless wall at the other end of the plank. Find the horizontal displacement of the plank after all the snowballs have hit the wall. The ant and the plank were both at rest at the start.

***Solution*** There is no net external force on the system of person and plank. So its center of mass does not move (it starts at rest; Section 4.5.5). Put the origin at the center of mass of the plank before the person starts to move. The person is then at $\frac{l}{2}$, so the center of mass of the whole system is at $\frac{ml}{2(m+M)}$. After the person reaches the other end, let the center of mass of the plank be at $x$; the person is then at $x - \frac{l}{2}$. So

$$\frac{m\left(x - \frac{l}{2}\right) + Mx}{m + M} = \frac{ml}{2(m + M)} \implies x = \frac{ml}{m + M}.$$

This is also easy to see another way: the final state is the initial state flipped horizontally about the center of mass. So the displacement is twice the initial coordinate of the center of mass, $\frac{ml}{2(m+M)} \times 2 = \frac{ml}{m+M}$.

The second problem is in effect the same as the first: a mass $m$ is moved from one end of the plank to the other. So the answer is the same.

**11. Pulling a Block*** [CP1 P4.11]

You and a block of metal are at rest on frictionless horizontal ground, a distance $l$ apart. Your mass is $m$ and the block's mass is $M$. You pull the block with a constant force $F$ through a massless string. Find the time $t$ at which you collide with the block.

***Solution*** The block's acceleration is $a_b = \frac{F}{M}$. By Newton's third law, the string also pulls you with force $F$, so your acceleration is $a_p = \frac{F}{m}$. The two accelerations point in opposite directions (towards each other). So in your frame, the block effectively accelerates towards you at

$$a'_b = F\left(\frac{1}{M} + \frac{1}{m}\right).$$

Both start at rest, so by the kinematics equation

$$l = \frac{1}{2}a'_b t^2 \implies t = \sqrt{\frac{2lMm}{F(M+m)}}.$$

**12. Falling Slinky*** [CP1 P4.12]

A slinky with negligible relaxed length, mass $m$ and spring constant $k$ is held vertically in mid-air by its top. The top is released, so it starts to fall and collide with the other parts of the slinky. The parts that have not yet been hit by the top are seen to stay still. Find the time, measured from the release, at which the bottom of the slinky starts to move. Repeat the calculation if an extra mass $m$ hangs from the bottom of the slinky at the start.

***Solution*** Put the origin at the top of the slinky, with the $y$-axis positive downwards.

*First case.* From the slinky example in the main text (with relaxed length $l = 0$), the bottom of the slinky is at $y = \frac{mg}{2k}$ and its center of mass is at $y = \frac{mg}{3k}$. After release, the only external force on the slinky is its weight $mg$. So its center of mass accelerates from rest at $g$. The bottom starts to move only when the center of mass reaches the bottom end, that is, when the top has collected all the other parts. This means the center of mass travels $\frac{mg}{2k} - \frac{mg}{3k} = \frac{mg}{6k}$. By basic kinematics,

$$\frac{1}{2}gt^2 = \frac{mg}{6k} \implies t = \sqrt{\frac{m}{3k}}.$$

*Second case.* The extra weight stretches the whole slinky uniformly by an extra total length $\frac{mg}{k}$. So the bottom of the slinky is at $y = \frac{3mg}{2k}$, and its center of mass is at $y = \frac{mg}{3k} + \frac{mg}{2k} = \frac{5mg}{6k}$. The time needed is then given by

$$\frac{1}{2}gt^2 = \frac{3mg}{2k} - \frac{5mg}{6k} = \frac{2mg}{3k} \implies t = 2\sqrt{\frac{m}{3k}}.$$

NOTE *(added in rewrite, not in the original)* In the second case the slinky feels, besides its weight $mg$, the downward pull $mg$ of the hanging mass, which stays at rest (and so needs a supporting tension $mg$) until the collapse reaches it. With a net force $2mg$, the slinky's center of mass accelerates at $2g$, not $g$, which gives $\frac{1}{2}(2g)t^2 = \frac{2mg}{3k}$, that is, $t = \sqrt{\frac{2m}{3k}}$. The same value follows from the slinky-plus-mass system: net force $2mg$, total mass $2m$, center of mass moving from $\frac{7mg}{6k}$ to $\frac{3mg}{2k}$. CP's printed answer $2\sqrt{\frac{m}{3k}}$ uses acceleration $g$; check this step.

## Systems with Constraints

For the following problems, assume that there is no friction between any surfaces and that strings and pulleys have negligible mass, unless stated otherwise.

**13. Atwood's Machine 1*** [CP1 P4.13]

Find the accelerations of the masses $m$ and $M$ if the ramp does not move. There is friction between the ramp and $M$, with static coefficient $\mu_s$ and kinetic coefficient $\mu_k < \mu_s$.

Figure 4.37
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**Figure 4.37.** Atwood's machine 1 (problem figure).

***Solution*** *When the system stays at rest.* The forces on both $m$ and $M$ must balance. Let the friction on $M$ be $f$, pointing down the slope ($f$ may turn out negative). Then

$$\begin{aligned} T - mg = 0 &\implies T = mg, \\ T - Mg\sin\theta - f = 0 &\implies f = mg - Mg\sin\theta. \end{aligned}$$

The normal force on $M$ from the slope is $N = Mg\cos\theta$. The friction must not be more than the largest possible static friction:

$$\begin{aligned} |f| &\le \mu_s N = \mu_s Mg\cos\theta, \\ |mg - Mg\sin\theta| &\le \mu_s Mg\cos\theta. \end{aligned}$$

This needs $M\sin\theta \le m \le M(\mu_s\cos\theta + \sin\theta)$ or $M(\sin\theta - \mu_s\cos\theta) \le m \le M\sin\theta$.

*When the system moves.* If $m > M(\mu_s\cos\theta + \sin\theta)$ or $m < M(\sin\theta - \mu_s\cos\theta)$, the system cannot stay at rest. Let the acceleration of $M$ along the plane be $a$, and the vertical acceleration of $m$ be $a_y$, both positive upwards. $F = ma$ for the blocks, and the conservation of string, give

$$\begin{aligned} ma_y &= T - mg, \\ Ma &= T - Mg\sin\theta - f, \\ a &= -a_y. \end{aligned}$$

Here $f$ is either $\mu_k N = \mu_k Mg\cos\theta$ or $-\mu_k N = -\mu_k Mg\cos\theta$, depending on which way $M$ slides relative to the slope. For now we keep the symbol $f$. Subtracting the second equation from the first and using $a = -a_y$,

$$\begin{aligned} a_y &= \frac{Mg\sin\theta - mg}{m+M} + \frac{f}{m+M}, \\ a &= \frac{mg - Mg\sin\theta}{m+M} - \frac{f}{m+M}. \end{aligned}$$

To find the sign of $f$, look at the sign of $a$ without friction.

- If $m > M(\mu_s\cos\theta + \sin\theta)$, $M$ moves and $a$ without friction is positive. Friction must then reduce this positive $a$, so $f$ takes the positive value $\mu_k Mg\cos\theta$.
- If $m < M(\sin\theta - \mu_s\cos\theta)$, $M$ moves and $a$ without friction is negative. Friction must then make $a$ less negative, so $f$ takes the negative value $-\mu_k Mg\cos\theta$.

**14. Atwood's Machine 2*** [CP1 P4.14]

Find the force $F$ needed to prevent any relative motion of $m$, $M$ and $\mu$.

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**Figure 4.38.** Atwood's machine 2 (problem figure).

***Solution*** Let the tension in the string be $T$. For $m$ to have no vertical motion relative to the others, $F = ma$ for $m$ in the vertical direction gives

$$T - mg = 0 \implies T = mg.$$

With no relative motion, all three blocks have the same acceleration $a_x$. So for the three blocks as one system,

$$F = (M + \mu + m)a_x.$$

Finally, $F = ma$ for mass $M$ gives

$$T = Ma_x \implies a_x = \frac{mg}{M},$$

and so

$$F = \frac{m(M + \mu + m)g}{M}.$$

**15. Traveling Together**** [CP1 P4.15]

Find the largest value of $m_1$ in the set-up below for which the masses $m_2$ and $m_3$ stay at rest relative to each other. The coefficient of kinetic friction between $m_2$ and the table is $\mu_k$, and the coefficient of static friction between $m_2$ and $m_3$ is $\mu_s$. Assume that $m_2$ moves relative to the table.

Figure 4.39
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**Figure 4.39.** Traveling together (problem figure).

***Solution*** Let the tension in the string be $T$, the friction between the table and $m_2$ be $f_1$, the friction between $m_2$ and $m_3$ be $f_2$, and the common acceleration of the masses be $a$. (Using one common $a$ is the conservation of string, used here without saying so.) Take rightwards and downwards as positive.

For the system containing $m_1$ alone,

$$m_1 a = m_1 g - T.$$

For the system containing $m_2$ and $m_3$,

$$(m_2 + m_3)a = T - f_1.$$

Adding these, and using $f_1 = \mu_k(m_2 + m_3)g$ (the table supports the weight of $m_2$ and $m_3$),

$$(m_1 + m_2 + m_3)a = m_1 g - f_1 = m_1 g - \mu_k(m_2 + m_3)g$$
$$a = \frac{m_1 - \mu_k(m_2 + m_3)}{m_1 + m_2 + m_3} g.$$

Next, isolate $m_3$ alone. Only the static friction $f_2$ accelerates it:

$$|f_2| = m_3 a \le \mu_s m_3 g$$
$$\implies \quad \frac{m_1 - \mu_k(m_2 + m_3)}{m_1 + m_2 + m_3} \le \mu_s.$$

We use $|f_2| = m_3 a$ and not $|f_2| = -m_3 a$ because $a > 0$. Indeed, $m_1 > \mu_0(m_2 + m_3) > \mu_k(m_2 + m_3)$ is needed for the set-up to move at all, where $\mu_0$ is the coefficient of static friction between the table and $m_2$. Solving,

$$m_1 \le \frac{(\mu_s + \mu_k)(m_2 + m_3)}{1 - \mu_s}.$$

**16. Sliding down a Plane**** [CP1 P4.16]

A block of mass $m$ is held at rest on a frictionless plane of mass $M$ and angle of inclination $\theta$. There is no friction between the inclined plane and the ground. The block is released. What is the horizontal acceleration of the plane?

***Solution*** Take rightwards and upwards as positive. Let $N$ be the normal force on the block from the plane (this is not necessarily $mg\cos\theta$), $a_x$ and $a_y$ the horizontal and vertical accelerations of the block, and $A$ the horizontal acceleration of the plane. The $F = ma$ equations in the vertical and horizontal directions are

$$N\cos\theta - mg = ma_y,$$
$$N\sin\theta = ma_x,$$
$$-N\sin\theta = MA.$$

(The last equation is for the plane: by the third law, the block pushes on the plane with the reverse of $N$.) For the block to stay on the plane, as in the main text,

$$\frac{a_y}{a_x - A} = -\tan\theta.$$

Putting $a_x = \frac{N\sin\theta}{m}$, $a_y = \frac{N\cos\theta}{m} - g$ and $A = -\frac{N\sin\theta}{M}$ into the constraint and solving,

$$N = \frac{g}{\sin\theta\tan\theta\left(\frac{1}{m} + \frac{1}{M}\right) + \frac{\cos\theta}{m}},$$
$$A = -\frac{N\sin\theta}{M} = -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}.$$

**17. Atwood's Machine 3**** [CP1 P4.17]

Find all tensions and the accelerations of the masses in the set-up below.

**Figure 4.40**

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FIGURE 4.40. Atwood's machine 3 (problem figure).

***Solution*** If the tension in the string holding $m_1$ is $T$, the tension in the string holding $m_2$ is $4T$. So the equations of motion are

$$m_1 a_1 = m_1 g - T,$$
$$m_2 a_2 = m_2 g - 4T,$$

with downwards positive. For the conservation of string equation, imagine that the second and fourth pulleys at the top (counting from the left) move down by $a$ and $b$. Let the displacements of $m_1$ and $m_2$ be $x_1$ and $x_2$. Then

$$x_1 + 2a + 2b = 0,$$

since the second and fourth pulleys take up a length of string equal to twice the distance they move. Also, from the piece of string holding up the pulley attached to $m_2$,

$$a + b = 2x_2,$$

since moving $m_2$ down by $x_2$ needs an extra $2x_2$ of string. Combining,

$$4x_2 + x_1 = 0 \quad \implies \quad a_1 + 4a_2 = 0.$$

Here $a$ and $b$ are displacements of pulleys, not accelerations. Solving (multiply the equations by $m_2$ and $4m_1$ and add),

$$T = \frac{5m_1 m_2 g}{16m_1 + m_2},$$
$$a_1 = \frac{16m_1 - 4m_2}{16m_1 + m_2} g,$$
$$a_2 = \frac{m_2 - 4m_1}{16m_1 + m_2} g.$$

**18. Atwood's Machine 4**** [CP1 P4.18]

Find the accelerations of the masses $m$ and $2m$. For the two pulleys shown, one continuous string goes once around the bottom pulley, once around the top pulley and once more around the bottom pulley, and is then tied to the ceiling.

**Figure 4.41**

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FIGURE 4.41. Atwood's machine 4 (problem figure).

***Solution*** The tension is the same all along the string; call it $T$. Four segments of the string hold up the right mass. With downwards positive, the $F = ma$ equations for the two masses are

$$ma_1 = mg - T,$$
$$2ma_2 = 2mg - 4T.$$

For the conservation of string: if the right mass moves by a distance $d$, the left mass must move $4d$ the other way, because four string segments are joined to the right mass but only one to the left mass. So

$$a_1 + 4a_2 = 0.$$

To find $T$, add the first equation to twice the second:

$$m(a_1 + 4a_2) = 5mg - 9T \quad \Longrightarrow \quad T = \frac{5}{9}mg,$$

$$a_1 = \frac{4}{9}g, \qquad a_2 = -\frac{1}{9}g.$$

**19. Atwood's Machine 5**** [CP1 P4.19]

Find the accelerations of all the masses in the set-up below. Try to get the conservation of string equation by picturing how the strings actually move.

**Figure 4.42**

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FIGURE 4.42. Atwood's machine 5 (problem figure).

***Solution*** Number the masses from left to right. The $F = ma$ equations are

$$ma_1 = mg - T,$$
$$ma_2 = mg - T,$$
$$ma_3 = mg - 2T.$$

**Figure 4.43**

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FIGURE 4.43. Atwood's machine 4 (CP's caption for this figure).

To get the conservation of string equation, suppose the middle mass moves up, above the dotted line, by a distance $d$ (Fig. 4.43). This frees a length $2d$ of string elsewhere. But a length $d$ must be "added" to the string joining the middle mass and the bottom pulley (above the dotted line). So only an extra length $d$ is left to be shared between the pulleys of the other two masses. Let $x$ and $y$ be the lengths of string gained by the pulleys of the left and right masses. Then the right mass only moves down by $\frac{y}{2}$, because the string on each side of its pulley must get longer by $\frac{y}{2}$. We can then write the accelerations of the masses as second time derivatives of $x$, $y$ and $d$:

$$x + y = d, \qquad a_2 = -\ddot{d}, \qquad a_1 = \ddot{x}, \qquad a_3 = \frac{\ddot{y}}{2},$$
$$\Longrightarrow \quad a_1 + a_2 + 2a_3 = 0.$$

Putting the $F = ma$ equations into this gives $4g - \frac{6T}{m} = 0$. Solving,

$$T = \frac{2}{3}mg, \quad a_1 = \frac{1}{3}g, \quad a_2 = \frac{1}{3}g, \quad a_3 = -\frac{1}{3}g.$$

**20. Atwood's Machine 6**** [CP1 P4.20]

Find the accelerations of all the masses in the set-up below.

**Figure 4.44**

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FIGURE 4.44. Atwood's machine 6 (problem figure).

***Solution*** Number the masses from left to right. The key point in this problem is that the tension in the string above the middle mass need not equal the tension in the string below it, because they are separate strings. Let the tension above the middle mass be $T_1$ and the tension below it be $T_2$. Then

$$ma_1 = mg - T_1,$$
$$ma_2 = mg + T_2 - T_1,$$
$$ma_3 = mg - 2T_2.$$

Here we have two conservation of string equations, one for each string:

$$a_2 = -a_1, \qquad 2a_3 = a_2.$$

Solving (the first string equation gives $T_1 = mg + \frac{T_2}{2}$; the second then gives $T_2$),

$$T_1 = \frac{11}{9}mg, \qquad T_2 = \frac{4}{9}mg,$$

$$a_1 = -\frac{2}{9}g, \qquad a_2 = \frac{2}{9}g, \qquad a_3 = \frac{1}{9}g.$$

**21. Pulling a Mass**** [CP1 P4.21]

A ball of mass $m$ is tied to two strings. One, of length $l_1$, is tied to a fixed pivot A. The other passes around a fixed pulley B and is pulled at constant velocity $v$. The length of the segment between the mass and B at this moment is $l_2$. In terms of the angles $\alpha$ and $\beta$ marked in the diagram below, find the tension on the ball from the second string.

Figure 4.45

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FIGURE 4.45. Pulling a mass (problem figure).

***Solution*** Since pivot A is fixed, the mass can only move perpendicular to the first string (tangentially). Let this tangential speed be $u$, anticlockwise.

Figure 4.46

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FIGURE 4.46. Velocity and acceleration of the mass.

The component of the ball's velocity along the second string is the rate at which the segment from the mass to the fixed pulley B gets shorter: $-\dot{l}_2 = v$ (think of polar coordinates centered on B). So

$$u\cos(90^\circ - \alpha - \beta) = v$$

$$u = \frac{v}{\sin(\alpha + \beta)}.$$

Next, let the radial and tangential accelerations of the mass relative to the first string be $a_r$ and $a_\theta$, as marked in Fig. 4.46. Since A is fixed and the first string keeps its length, $a_r$ must be the centripetal acceleration:

$$a_r = \frac{u^2}{l_1} = \frac{v^2}{l_1 \sin^2(\alpha + \beta)}.$$

Similarly, $\ddot{l}_2 = 0$ (the string is pulled at constant speed). So in the radial equation of polar coordinates about B, the component of acceleration along the second string must also be the centripetal acceleration that goes with the velocity $u\cos(\alpha + \beta)$ perpendicular to the second string:

$$a_\theta \sin(\alpha + \beta) - a_r \cos(\alpha + \beta) = \frac{u^2 \cos^2(\alpha + \beta)}{l_2}.$$

Substituting the expressions for $u$ and $a_r$,

$$a_\theta = \boxed{\frac{v^2 \cos(\alpha + \beta)}{\sin^3(\alpha + \beta)} \left( \frac{1}{l_1} + \frac{\cos(\alpha + \beta)}{l_2} \right)}.$$

Finally, look at the forces on the mass. Let the tension on the mass from the second string be $T_2$; the tension from the first string does not matter here, because we take components perpendicular to the first string. Taking the components of $T_2$ and of the weight perpendicular to the first string,

$$T_2 \sin(\alpha + \beta) - mg\cos\alpha = ma_\theta$$

$$\Longrightarrow \boxed{\begin{aligned} T_2 = {}& \frac{mg\cos\alpha}{\sin(\alpha + \beta)} \\ &+ \frac{mv^2 \cos(\alpha + \beta)}{\sin^4(\alpha + \beta)} \left( \frac{1}{l_1} + \frac{\cos(\alpha + \beta)}{l_2} \right) \end{aligned}}.$$

**22. Equivalent Mass**** [CP1 P4.22]

Two masses $m_1$ and $m_2$ are connected over a pulley. Suppose we can replace this set-up by an equivalent one made of a single mass $m_{\text{eq}}$ hanging from a string. Find $m_{\text{eq}}$. Hint: if the set-ups are equivalent, any system connected to them should behave in the same way.

***Solution*** The pulley and the equivalent mass must have equal accelerations, so that the string holding them is conserved in the same way for every outside set-up joined to them. Call this common acceleration $a_{\text{eq}}$. Also, for the set-ups to be fully equivalent, the tensions in the strings holding the pulley and the equivalent mass must be the same. Call this common tension $T$; the string over the pulley then has tension $\frac{T}{2}$.

Let the accelerations of $m_1$ and $m_2$ be $a_{\text{eq}} + a'$ and $a_{\text{eq}} - a'$ (their average must be $a_{\text{eq}}$). Then

$$\begin{aligned} m_1 g - \frac{T}{2} &= m_1(a_{\text{eq}} + a'), \\ m_2 g - \frac{T}{2} &= m_2(a_{\text{eq}} - a'), \\ m_{\text{eq}} g - T &= m_{\text{eq}} a_{\text{eq}}. \end{aligned}$$

Solving (divide the first two by $m_1$ and $m_2$ and add to remove $a'$),

$$m_{\text{eq}} = \frac{4m_1m_2}{m_1+m_2},$$
$$T = \frac{4m_1m_2(g-a_{\text{eq}})}{m_1+m_2},$$
$$a' = \frac{m_1-m_2}{m_1+m_2}(g-a_{\text{eq}}).$$

Note that $a_{\text{eq}}$, and hence $T$ and $a'$, cannot be found here: they depend on the set-up that these masses are connected to.

**23. Infinite Atwood Machine***** [CP1 P4.23]

Infinitely many identical masses are arranged as shown below. Find the acceleration of every mass. Assume the set-up has $N$ masses, with the $N$th mass taking the place of what would have been the $N$th pulley, and then let $N \to \infty$. What would happen if the $N$th mass were a massless pulley instead? The result of the previous problem may be useful. (Adapted from "Introduction to Classical Mechanics.")

**Figure 4.47**

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FIGURE 4.47. Infinite Atwood machine (problem figure).

***Solution*** *(a) Method 1: equivalent masses.* The previous problem showed that the equivalent mass is

$$m_{\text{eq}} = \frac{4m_1m_2}{m_1+m_2}.$$

Apply this from the bottom up. At each step $m_1$ stays equal to $m$, while $m_2$ is the equivalent mass found so far. Since $m_2$ starts at $m$, we are in effect applying the function

$$f(x) = \frac{4mx}{m+x}$$

to $m$ infinitely many times, one layer after another. Notice that $f(x) > x$ whenever $x < 3m$, and that $f(3m) = 3m$ is a fixed point. So, starting from $m$, each step increases the equivalent mass towards $3m$. The whole arrangement to the right of the top pulley is therefore equivalent to a mass $3m$. This reduces the problem to the simplest Atwood machine with two masses, which we solved earlier. Numbering the masses $1, 2, \ldots$ from left to right, the result of the first example in Section 4.8.3 (with $m_1 = m$, $m_2 = 3m$) gives

$$a_1 = -\frac{g}{2},$$

with downwards positive. Let the tension on the first mass be $T$. Then the tension on the $k$th mass is $\frac{T}{2^{k-1}}$ (each lower pulley halves the tension). $F = ma$ for the first and the $k$th mass gives

$$mg - T = ma_1 \implies T = \frac{3mg}{2},$$
$$mg - \frac{T}{2^{k-1}} = ma_k \implies a_k = \left(1 - \frac{3}{2^k}\right)g.$$

*(b) Method 2: scaling and symmetry.* Again let the tension on the first mass be $T$. Then the tension in the string joining the first pulley to the wall is $2T$, and the tension in the string joining the second pulley to the first mass is $T$. If the system of masses is fixed, the tension $2T$ between the first pulley and the wall can only depend on the gravitational field strength $g$. By dimensional analysis it must be proportional to $g$:

$$\frac{2T}{g} = k$$

for some constant $k$, as long as we keep the same system of masses, even if the field strength changes.

Now look at the system of masses on the right of the first pulley. The second pulley accelerates at $-a_1$, with $a_1$ positive downwards. So in the frame of the second pulley, it is as if the masses below it lived in a world where gravity is $g + a_1$. Also, the tension in the string holding the second pulley plays the same role as the tension in the string joining the first pulley to the wall, because the second pulley "does not know" what it is attached to. Using this symmetry, and the fact that the tension depends only on the apparent gravitational field strength,

$$\frac{2T}{g} = \frac{T}{g+a_1}.$$

This gives at once

$$a_1 = -\frac{g}{2}.$$

Using the $F = ma$ equations for the first and the $k$th mass again gives

$$a_k = \left(1 - \frac{3}{2^k}\right)g.$$

*If the last mass is a massless pulley.* $T = 0$ is also a solution of the equation above, but it means that all the masses are in free fall. This is what happens if the last mass is a massless pulley: the tension in the last string is then zero, which makes the tension in every string zero. Method 1 gives the same conclusion, since the equivalent mass is then clearly zero. So a single mass can have a surprisingly large effect!

## Polar Coordinates/Circular Motion

**24. Rotating Rod**** [CP1 P4.27]

A uniform rod of linear mass density $\lambda$ and length $l$ lies on a horizontal table. Use polar coordinates with the origin at one end of the rod. The rod rotates about the origin at a constant angular velocity $\omega$. Find the tension as a function of the radial coordinate, $T(r)$, (i) if the end at $r = 0$ is free (the end at $r = l$ could be attached to a rotating cylinder, for example), and (ii) if the end at $r = l$ is free (the end at $r = 0$ could be skewered and turned, for instance).

***Solution*** Take a tiny piece between $r$ and $r + \mathrm{d}r$, with outwards positive. The tension $T$ at $r$ pulls the piece radially inwards, while the tension at $r + \mathrm{d}r$, written $T + \mathrm{d}T$, pulls it radially outwards. The net radial force $\mathrm{d}T$ must supply the centripetal force on the piece, whose mass is $\lambda\,\mathrm{d}r$ and whose acceleration is $-r\omega^2$:

$$\mathrm{d}T = -\lambda r\omega^2\,\mathrm{d}r \quad\Longrightarrow\quad T(r) = -\frac{\lambda r^2\omega^2}{2} + c,$$

where $c$ is fixed by a boundary condition. The tension at a free end must be zero. So if $r = 0$ is a free end, $c = 0$:

$$T(r) = -\frac{\lambda r^2\omega^2}{2}.$$

If $r = l$ is a free end, $c = \frac{\lambda l^2\omega^2}{2}$:

$$T(r) = \frac{\lambda\omega^2}{2}\left(l^2 - r^2\right).$$

**Note** *(added in rewrite, not in the original)* A negative $T$ means the rod is being pushed together (compressed) rather than pulled apart. This is possible for a rigid rod, though not for a string.

**25. Rotating Chain**** [CP1 P4.28]

A chain of uniform linear mass density $\lambda$ forms a circle of radius $R$ and is wrapped around a frictionless cone with half angle $\alpha$. The chain rotates about the symmetry axis of the cone at a constant angular velocity $\omega$, while keeping its shape as a circle of radius $R$. Find the tension $T$ in the chain.

***Solution*** Take a tiny piece of chain of length $R\,\mathrm{d}\theta$, between $\theta$ and $\theta + \mathrm{d}\theta$ in polar coordinates in the plane of the chain. Figure 4.48 shows its free-body diagram.

**Figure 4.48**

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**Figure 4.48.** Free-body diagram of an infinitesimal segment.

$T$ is the tension at the two ends of this piece, from the neighboring pieces (the two must be equal so that the piece has no tangential acceleration). $\mathrm{d}N$ is the normal force on the piece from the cone. Because the cone's surface is slanted, $\mathrm{d}N$ points at the angle $\alpha$ above the plane of the chain. $\mathrm{d}W$ is the weight of the piece, $\lambda Rg\,\mathrm{d}\theta$. For the vertical forces to balance,

$$\mathrm{d}N\sin\alpha = \mathrm{d}W \quad\Longrightarrow\quad \mathrm{d}N = \frac{\lambda Rg}{\sin\alpha}\,\mathrm{d}\theta\,.$$

The two tensions each make a small angle $\frac{\mathrm{d}\theta}{2}$ with the tangent, so together they have an inward radial component $2T\sin\frac{\mathrm{d}\theta}{2} \approx T\,\mathrm{d}\theta$, using $\sin x \approx x$ for small $x$. The net radial force (outwards positive) is then

$$F_r = \mathrm{d}N\cos\alpha - T\,\mathrm{d}\theta = (\lambda Rg\cot\alpha - T)\,\mathrm{d}\theta\,.$$

This must equal the required centripetal force (mass $\lambda R\,\mathrm{d}\theta$ times $-R\omega^2$), so

$$(\lambda Rg\cot\alpha - T)\,\mathrm{d}\theta = -\lambda R\,\mathrm{d}\theta\ R\omega^2$$
$$T = \lambda R^2\omega^2 + \lambda Rg\cot\alpha\,.$$

## Systems with Varying Amounts of Moving Mass

**26. Sweeping Pan**** [CP1 P4.29]

A sweeping pan of width $l$ and initial mass $M$ is at rest on a horizontal, frictionless table. Dust is spread evenly over the whole table, with surface mass density $\sigma$. Find $F(t)$, the force needed to push the pan so that it moves with a constant acceleration $a$.

***Solution*** After the pan has moved a distance $x$, it has swept up an area $lx$ of dust, so its mass is

$$m(x) = M + \sigma lx.$$

Since the pan moves with constant acceleration $a$ from rest,

$$x = \frac{1}{2}at^2, \qquad v = at, \qquad m = M + \frac{1}{2}\sigma lat^2.$$

Take the system to be the pan and all the dust (the dust not yet swept is at rest). The momentum at time $t$ is

$$p = mv = Mat + \frac{1}{2}\sigma la^2t^3,$$
$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = Ma + \frac{3}{2}\sigma la^2t^2.$$

**27. Holding a Rope**** [CP1 P4.30]

Take a uniform rope of length $l$ and linear mass density $\lambda$. Hold it by its ends, vertically, so that it forms two segments of length $\frac{l}{2}$ with a small bend at the bottom (small in the same sense as in the falling-chain example: the bend is effectively massless). Now gently let go of one end while still holding the other. The parts of the rope that pass through the bend stop at once. Find $F(t)$, the force you exert on the rope as a function of time.

***Solution*** The key point is that there can be no tension in the rope. The small bend is essentially horizontal, and the tensions on both sides of it point upwards. Any tension would give the massless bend an infinite upward acceleration. So the moving part of the rope is in free fall.

After time $t$, the free end has dropped a vertical distance $\frac{gt^2}{2}$. But only a length $\frac{gt^2}{4}$ of rope has passed over the bend (as with a movable pulley: the bend moves down half as far as the free end). So a length $\frac{l}{2} - \frac{gt^2}{4}$ of rope is still moving, at velocity $gt$, with downwards positive. The momentum of the whole rope is therefore

$$p = \lambda \left( \frac{l}{2} - \frac{gt^2}{4} \right) gt.$$

The net force on the whole rope is its weight minus the force you exert. It must equal the rate of change of momentum:

$$\begin{aligned} \lambda g l - F &= \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\lambda g l}{2} - \frac{3\lambda g^2 t^2}{4} \\ F &= \frac{\lambda g l}{2} + \frac{3\lambda g^2 t^2}{4}. \end{aligned}$$

At $t = \sqrt{\frac{2l}{g}}$, $F$ suddenly drops from $2\lambda g l$ to $\lambda g l$ (the weight of the rope). From then on, there are no more downward-moving parts of the rope crossing the bend that must be given an upward acceleration to stop them on the other side.

**28. Pulling a Rope**** [CP1 P4.31]

A long rope of linear mass density $\lambda$ lies on a horizontal, frictionless table, with a small bend as shown in the figure (small as in the falling-chain example). You grab the end of the rope near the bend and start to pull it. At the start, the length of rope that has passed the bend is negligible, and the whole rope is at rest. (Adapted from "Introduction to Classical Mechanics".)

(1) Find the force $F$ needed to pull the rope so that it moves at a constant velocity $v$.

(2) Find the force $F(t)$ needed to pull the rope so that it moves with a constant acceleration $a$.

(3) Prove the reverse of part (1): if a constant force $F$ acts on the rope, show that it moves at a constant velocity.

(4) Now replace the force on the rope by a spring, so that the rightwards force on the rope is $k(L-x)$, where $L$ is a constant and $x$ is the rightwards displacement of the end the spring is attached to. Find $x(t)$, the displacement of this end as a function of time, for $x \le L$.

**Figure 4.49**

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**Figure 4.49.** Pulling a rope (problem figure).

***Solution*** At every moment, only the parts of rope that have passed the bend are moving, because the rope is slack everywhere (see the previous solution). After the end has been pulled a distance $x$, only a length $\frac{x}{2}$ of rope moves, at speed $\dot{x}$ (again as with a movable pulley). So the momentum of the whole rope is

$$p(x) = \frac{\lambda x \dot{x}}{2}.$$

The net force on the rope equals its rate of change of momentum:

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\lambda \dot{x}^2}{2} + \frac{\lambda x \ddot{x}}{2}.$$

*(1)* Here $\dot{x} = v$ and $\ddot{x} = 0$. The force needed is

$$F = \frac{\lambda v^2}{2}.$$

*(2)* Here $x = \frac{1}{2}at^2$, $\dot{x} = at$ and $\ddot{x} = a$. The force needed is

$$F = \frac{\lambda a^2 t^2}{2} + \frac{\lambda a^2 t^2}{4} = \frac{3\lambda a^2 t^2}{4}.$$

*(3)* $F$ is constant and the rope starts at rest, so $p = Ft$. Then

$$\begin{aligned} \frac{\lambda x}{2}\frac{\mathrm{d}x}{\mathrm{d}t} &= Ft \\ \int_0^x \frac{\lambda}{2} x \,\mathrm{d}x &= \int_0^t Ft \,\mathrm{d}t \\ \frac{\lambda x^2}{4} &= \frac{Ft^2}{2} \\ x &= \sqrt{\frac{2F}{\lambda}}\, t \\ \dot{x} &= \sqrt{\frac{2F}{\lambda}}, \end{aligned}$$

a constant velocity.

*(4)* Here $F = k(L-x)$. Using the trick $\ddot{x} = \frac{1}{2}\,\mathrm{d}\dot{x}^2/\mathrm{d}x$,

$$k(L-x) = \frac{\lambda \dot{x}^2}{2} + \frac{\lambda x}{4}\frac{\mathrm{d}\dot{x}^2}{\mathrm{d}x}.$$

Multiply by the integrating factor $4x$. The left side of the result is an exact derivative:

$$2\lambda x \dot{x}^2 + \lambda x^2 \frac{\mathrm{d}\dot{x}^2}{\mathrm{d}x} = \lambda \frac{\mathrm{d}(x^2 \dot{x}^2)}{\mathrm{d}x} = 4k(L-x)x.$$

Integrate, using $x = 0$ and $\dot{x} = 0$ at $t = 0$:

$$\lambda \int_0^{x^2\dot{x}^2} \mathrm{d}(x^2\dot{x}^2) = \int_0^x 4k(L-x)x\,\mathrm{d}x$$
$$\lambda x^2\dot{x}^2 = 2kLx^2 - \frac{4kx^3}{3}$$
$$\dot{x} = \sqrt{\frac{2kL}{\lambda} - \frac{4kx}{3\lambda}}.$$

We take the positive root, since $\dot{x} > 0$ for $0 < x \le L$, as $\ddot{x} \ge 0$. Write $\dot{x} = \sqrt{\frac{4k}{3\lambda}}\sqrt{\frac{3}{2}L - x}$ and separate variables:

$$\int_0^x \frac{1}{\sqrt{\frac{3}{2}L - x}}\,\mathrm{d}x = \int_0^t \sqrt{\frac{4k}{3\lambda}}\,\mathrm{d}t$$
$$\sqrt{6L} - \sqrt{6L - 4x} = \sqrt{\frac{4k}{3\lambda}}\,t.$$

(The left integral is $2\sqrt{\frac{3}{2}L} - 2\sqrt{\frac{3}{2}L - x}$, and $2\sqrt{\frac{3}{2}L} = \sqrt{6L}$.) So

$$6L - 4x = \left(\sqrt{6L} - \sqrt{\frac{4k}{3\lambda}}\,t\right)^2$$
$$x = \frac{3}{2}L - \left(\sqrt{\frac{3L}{2}} - \sqrt{\frac{k}{3\lambda}}\,t\right)^2.$$

This holds until $x = L$. After that, the force on the rope becomes compressive (the spring pushes), and the rope deforms.

**29. Drag Force on Sheet*** [CP1 P6.25]

A massive sheet is surrounded by sand at rest, with mass density $\rho$. The sheet moves at a constant speed $v$. Find the drag force per unit area on it, assuming the collisions are elastic.

***Solution*** Let the total area of the sheet be $A$. In time $\mathrm{d}t$, the sheet sweeps through a volume $Av\,\mathrm{d}t$, so it hits a mass $\mathrm{d}m = \rho Av\,\mathrm{d}t$ of sand. This sand gains momentum $2v\,\mathrm{d}m$, because it leaves the sheet at speed $2v$. (Find the change in velocity in the sheet's frame: there the sand comes in at $-v$ and, in an elastic bounce off the massive sheet, goes back out at $v$. Adding back the sheet's velocity $v$, the sand leaves at $2v$ in the lab frame.) The change in the sheet's momentum in this time is the negative of this. So

$$\mathrm{d}p = -2\rho Av^2\,\mathrm{d}t\,,$$
$$F_{\text{drag}} = \frac{\mathrm{d}p}{\mathrm{d}t} = -2\rho Av^2,$$
$$\frac{F_{\text{drag}}}{A} = -2\rho v^2.$$

**30. Propelling a Car**** [CP1 P6.26]

You throw baseballs at speed $u$ towards a car of mass $M$ that can move without friction on the ground. The baseballs leave your hand at a rate $\sigma$ (mass per unit time) and bounce elastically off the car's window, straight back. The car starts at rest. Find its speed and position as functions of time. ("An Introduction to Classical Mechanics")

***Solution*** Let the car's velocity at some moment be $v(t)$. Imagine you throw a mass $\mathrm{d}m$ of baseballs at the car. In the frame moving at $v$, the baseballs come towards the car at $(u - v)$ and bounce back at $(v - u)$. The change in the momentum of the baseballs, $(2v - 2u)\,\mathrm{d}m$, is the negative of the change in the car's momentum in the moving frame. So the change in the car's momentum in the moving frame is

$$\mathrm{d}p = 2(u-v)\,\mathrm{d}m\,,$$
$$F' = \frac{\mathrm{d}p}{\mathrm{d}t} = 2(u-v)\frac{\mathrm{d}m}{\mathrm{d}t} = F,$$

where $F'$ is the force on the car in the moving frame and $F$ is the force on the car in the lab frame. They are equal because forces are the same in all inertial frames under Galilean transformations (Note 3).

Note also that $\mathrm{d}m/\mathrm{d}t \neq \sigma$, where $\mathrm{d}m/\mathrm{d}t$ is the rate at which baseball mass hits the car. The baseballs leave your hand at velocity $u$, but they move at $(u - v)$ relative to the car. Think of it as something like the Doppler effect. In numbers: if neighboring baseballs are a distance $l$ apart, the time between hits is $\frac{l}{u-v}$, compared with $\frac{l}{u}$ if the car were at rest. So

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{u-v}{u}\sigma,$$
$$M\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{2(u-v)^2}{u}\sigma.$$

Separate variables and integrate:

$$\int_0^v \frac{1}{(u-v)^2}\,\mathrm{d}v = \frac{2\sigma}{Mu}\int_0^t \mathrm{d}t$$
$$\frac{1}{u-v} - \frac{1}{u} = \frac{2\sigma t}{Mu}$$
$$v = u - \frac{u}{1 + \frac{2\sigma t}{M}}.$$

Integrating once more for the position,

$$\int_0^x \mathrm{d}x = \int_0^t \left(u - \frac{u}{1 + \frac{2\sigma t}{M}}\right)\mathrm{d}t$$
$$x = ut - \frac{Mu}{2\sigma}\ln\left(1 + \frac{2\sigma t}{M}\right).$$

**31. Drag Force on Sphere**** [CP1 P6.27]

A massive sphere of radius $R$ is surrounded by air particles, which do not interact with each other, of mass density $\rho$. The sphere moves at speed $v$. Find the drag force on it, assuming the collisions are elastic.

***Solution*** Let the sphere move at velocity $v$ in the positive $y$-direction. In the sphere's frame, the incoming particles move at velocity $-v$. Look at a collision between a particle and a tiny surface element of the sphere at azimuthal angle $\varphi$ and at angle $\theta$ from the $z$-axis, in spherical coordinates. In an elastic bounce off the massive sphere, the particle's velocity component along the area vector of the surface element (which points radially outwards) is reversed; the other components do not change. So the particle's change in velocity is twice the negative of that component. The unit area vector is

$$\hat{\boldsymbol{r}} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}.$$

So the component of the particle's initial velocity along it is

$$\begin{pmatrix} 0 \\ -v \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = -v\sin\theta\sin\varphi.$$

The change in the particle's velocity is $-2$ times this, along the radial direction:

$$\Delta\vec{\boldsymbol{v}} = 2v\sin\theta\sin\varphi\,\hat{\boldsymbol{r}}.$$

The change in momentum of the particle, of mass $\mathrm{d}m$, is then

$$\mathrm{d}m\ \Delta\vec{\boldsymbol{v}} = 2v\sin\theta\sin\varphi\,\hat{\boldsymbol{r}}\cdot\mathrm{d}m\,.$$

By conservation of momentum, the change in the sphere's momentum is the negative of this:

$$\mathrm{d}\vec{\boldsymbol{p}} = -2v\sin\theta\sin\varphi\,\hat{\boldsymbol{r}}\cdot\mathrm{d}m\,.$$

Now we only need to find how much mass hits the surface element at $\varphi$, $\theta$ in a time $\mathrm{d}t$. Back in the lab frame, the volume swept by this surface element in time $\mathrm{d}t$ is the dot product of the sphere's velocity with the area vector of the element, times $\mathrm{d}t$. The area of the element is $\mathrm{d}A = R^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta$ (Note 1). So

$$\begin{aligned}\mathrm{d}m &= \rho\,\vec{\boldsymbol{v}}\cdot\mathrm{d}A\ \hat{\boldsymbol{r}}\ \mathrm{d}t \\ &= \rho\begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}\cdot\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}R^2\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}t \\ &= \rho v R^2\sin^2\theta\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}t\,.\end{aligned}$$

Hence

$$\frac{\mathrm{d}\vec{\boldsymbol{p}}}{\mathrm{d}t} = -2\rho v^2R^2\sin^3\theta\sin^2\varphi\,\hat{\boldsymbol{r}}\,\mathrm{d}\varphi\,\mathrm{d}\theta\,.$$

By symmetry, the total drag force on the sphere points only along the $y$-direction. So we only need the $y$-component of the above, using $\hat{r}_y = \sin\theta\sin\varphi$:

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = -2\rho v^2R^2\sin^4\theta\sin^3\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,.$$

The total drag force comes from integrating this over the half of the sphere's surface that faces forward (only that half is hit by particles), that is, $0 \le \varphi \le \pi$ and $0 \le \theta \le \pi$. Using $\int_0^\pi \sin^4\theta\,\mathrm{d}\theta = \frac{3\pi}{8}$ and $\int_0^\pi \sin^3\varphi\,\mathrm{d}\varphi = \frac{4}{3}$,

$$F_{\text{drag}} = -\int_0^\pi\int_0^\pi 2\rho v^2R^2\sin^4\theta\sin^3\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta = -\pi\rho v^2R^2.$$

Interestingly, comparing this with the drag force on a sheet (Problem 29) shows that the "effective area" of a sphere is half its cross-sectional area, $\frac{1}{2}\pi R^2$.

**32. Sweeping Duster***** [CP1 P6.29]

A rod-shaped duster has length $l$ and initial mass $M$. It is held at rest at the top of an inclined plane with angle of inclination $\theta$. The whole plane is covered with dust of surface mass density $\sigma$. The duster is given a slight push and picks up dust as it moves. Find its velocity $v$ along the plane as a function of $x$, the distance it has travelled along the surface of the plane.

***Solution*** The total mass of the duster plus the dust it has collected is $m = M + \sigma l x$. At any moment, the net force on the duster-plus-dust system along the plane is $mg\sin\theta$. Apply the impulse-momentum theorem over a time $\mathrm{d}t$ (the dust about to be picked up is at rest):

$$\begin{aligned} mg\sin\theta\,\mathrm{d}t &= (m+\mathrm{d}m)(v+\mathrm{d}v) - mv \\ \implies\quad mg\sin\theta &= \frac{\mathrm{d}m}{\mathrm{d}t}v + m\frac{\mathrm{d}v}{\mathrm{d}t}.\end{aligned}$$

Notice that

$$\frac{\mathrm{d}m}{\mathrm{d}t}v + m\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}v + m\dot{v} = \sigma l v^2 + m\dot{v}.$$

Hence

$$(M+\sigma l x)g\sin\theta = \sigma l v^2 + (M+\sigma l x)\dot{v}.$$

Let $y = \frac{M}{\sigma l} + x$ (so $\dot{y} = v$ and $\ddot{y} = \dot{v}$), and divide by $\sigma l$:

$$yg\sin\theta = \dot{y}^2 + y\ddot{y}.$$

Using $\ddot{y} = \frac{1}{2}\,\mathrm{d}\dot{y}^2/\mathrm{d}y$,

$$\dot{y}^2 + \frac{1}{2}y\frac{\mathrm{d}\dot{y}^2}{\mathrm{d}y} = yg\sin\theta.$$

Multiply by the integrating factor $2y$:

$$2y\dot{y}^2 + y^2\frac{\mathrm{d}\dot{y}^2}{\mathrm{d}y} = \frac{\mathrm{d}(y^2\dot{y}^2)}{\mathrm{d}y} = 2y^2g\sin\theta.$$

At the start, $y = \frac{M}{\sigma l}$ and $\dot{y} = 0$, so

$$\begin{aligned}\int_0^{y^2\dot{y}^2}\mathrm{d}(y^2\dot{y}^2) &= \int_{\frac{M}{\sigma l}}^{y} 2y^2g\sin\theta\,\mathrm{d}y \\ y^2\dot{y}^2 &= \frac{2}{3}y^3g\sin\theta - \frac{2M^3}{3\sigma^3l^3}g\sin\theta \\ \dot{y} &= \sqrt{\frac{2}{3}yg\sin\theta - \frac{2M^3g\sin\theta}{3\sigma^3l^3y^2}},\end{aligned}$$

and, going back to $x$,

$$\dot{x} = \left[ \frac{2}{3}\left(\frac{M}{\sigma l} + x\right) g \sin\theta - \frac{2M^3 g \sin\theta}{3\sigma^3 l^3 \left(\frac{M}{\sigma l} + x\right)^2} \right]^{1/2}.$$

**33. Raindrop***** [CP1 P6.30]

Model a raindrop as a blob that always keeps the shape of a uniform sphere of constant density. A raindrop, at first of negligible size, starts to fall through a uniform cloud of tiny water droplets. The raindrop collects the droplets it meets. Find its acceleration.

***Solution*** Let $\rho$ be the density of the raindrop and $\lambda$ the mass density of the droplets in the cloud (mass per unit volume of cloud), both uniform. Let $r(t)$, $m(t)$ and $v(t)$ be the raindrop's radius, mass and velocity at time $t$. We need three equations for these three unknowns: two expressions for $\dot{m}$ and the impulse-momentum theorem.

First, the only force on the raindrop is gravity. The droplets it collects are at rest. Taking downwards as positive, over a time d$t$,

$$\begin{aligned} mg\,\mathrm{d}t &= (m + \mathrm{d}m)(v + \mathrm{d}v) - mv \\ mg &= \dot{m}v + m\dot{v}. \end{aligned}$$

Next, since the raindrop is always a sphere,

$$m = \frac{4}{3}\pi r^3 \rho \quad \Longrightarrow \quad \dot{m} = 4\pi r^2 \dot{r} \rho.$$

We get another expression for $\dot{m}$ from the fact that in a time d$t$ the raindrop sweeps out a volume equal to its cross-sectional area times $v\,\mathrm{d}t$. So the mass gained in d$t$ is

$$\mathrm{d}m = \lambda\,\mathrm{d}V = \lambda\pi r^2 v\,\mathrm{d}t \quad \Longrightarrow \quad \dot{m} = \lambda\pi r^2 v.$$

Comparing the two expressions for $\dot{m}$,

$$v = \frac{4\rho}{\lambda}\dot{r}, \qquad \dot{v} = \frac{4\rho}{\lambda}\ddot{r}.$$

Put these into the equation from the impulse-momentum theorem:

$$\frac{4}{3}\pi r^3 \rho g = 4\pi r^2 \dot{r}\rho \cdot \frac{4\rho}{\lambda}\dot{r} + \frac{4}{3}\pi r^3 \rho \cdot \frac{4\rho}{\lambda}\ddot{r}.$$

Dividing by $\frac{4}{3}\pi r^2 \rho \cdot \frac{4\rho}{\lambda}$,

$$\frac{g\lambda}{4\rho}r = 3\dot{r}^2 + r\ddot{r}.$$

Using $\ddot{r} = \frac{1}{2}\,\mathrm{d}\dot{r}^2/\mathrm{d}r$,

$$3\dot{r}^2 + \frac{1}{2}r\frac{\mathrm{d}\dot{r}^2}{\mathrm{d}r} = \frac{g\lambda}{4\rho}r.$$

Multiply by the integrating factor $2r^5$. The left side becomes an exact derivative, $\mathrm{d}(r^6\dot{r}^2)/\mathrm{d}r$:

$$6r^5\dot{r}^2 + r^6\frac{\mathrm{d}\dot{r}^2}{\mathrm{d}r} = \frac{g\lambda}{2\rho}r^6.$$

The raindrop starts with negligible size, so

$$\int_0^{r^6\dot{r}^2} \mathrm{d}(r^6\dot{r}^2) = \int_0^r \frac{g\lambda}{2\rho}r^6\,\mathrm{d}r \quad \Longrightarrow \quad r^6\dot{r}^2 = \frac{g\lambda}{14\rho}r^7$$

$$\dot{r} = \sqrt{\frac{g\lambda}{14\rho}r}.$$

Separating variables again,

$$\int_0^r \frac{1}{\sqrt{r}}\,\mathrm{d}r = \int_0^t \sqrt{\frac{g\lambda}{14\rho}}\,\mathrm{d}t$$

$$2\sqrt{r} = \sqrt{\frac{g\lambda}{14\rho}}\,t \quad \Longrightarrow \quad r = \frac{g\lambda}{56\rho}t^2.$$

So $\ddot{r} = \frac{g\lambda}{28\rho}$, and the acceleration of the raindrop is

$$\dot{v} = \frac{4\rho}{\lambda}\ddot{r} = \frac{g}{7},$$

which does not depend on $\rho$, $\lambda$ or $t$.

*A faster, less rigorous way.* To solve

$$mg = \dot{m}v + m\dot{v},$$

guess from the question that $\dot{v}$ should be a constant $a$ (so $v = at$), since the question does not ask for the acceleration as a function of time. Since $v = \frac{4\rho}{\lambda}\dot{r}$, we get $r \propto t^2$, and then $m = \frac{4}{3}\pi r^3\rho$ gives $m = kt^6$ for some constant $k$. Substituting $m = kt^6$, $\dot{v} = a$ and $v = at$,

$$kt^6 g = 6kt^5 \cdot at + kt^6 a = 7kt^6 a \quad \Longrightarrow \quad a = \frac{g}{7}.$$

# FIGURE FILES FOR THIS CHAPTER

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